ISSN 097-3799

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

JOURNAL OF NATURAL & PHYSICAL SCIENCES

खण्ड 18 अंक 2, 2004
Vol. 18 No. 2, 2004

गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार
Gurukula Kangri Vishwavidyalaya, Haridwar

शोध पत्रिका पटल

अध्यक्ष	स्वतन्त्र कुमार कुलपति
उपाध्यक्ष	वीरेन्द्र अरोड़ा प्राचार्य
सचिव	ए.के. चोपड़ा कुलसचिव
सदस्यगण	एस.पी. सिंह वित्ताधिकारी
	वीरेन्द्र अरोड़ा प्रधान सम्पादक
	जे. पी. विद्यालंकार व्यवसाय प्रबन्धक
	पी.पी. पाठक प्रबंध संपादक

JOURNAL COUNCIL

President	**Swatantra Kumar** Vice Chancellor
Vice President	**Virendra Arora** Principal
Secretary	**A.K. Chopra** Registrar
Members	**S.P. Singh** Finance Officer
	Virendra Arora Chief Editor
	J. P. Vidyalankar Business Manager
	P. P. Pathak Managing Editor

परामर्शदाता मण्डल

एस.एल. सिंह (सेवा निवृत्त प्रोफेसर एवं प्राचार्य)
योल जे चो (ज्योंगसांग नेशनल वि0वि0, कोरिया)
एस. एन. मिश्रा (यूनिटरा, द. अफ्रीका)
सोम नैम्पल्ली (इमिरिटस् प्रोफेसर, कनाडा)
एस. पी. सिंह (मेमोरियल वि.वि., कनाडा)
देवकी एन. तलवार (इण्डियाना वि.वि., यू.एस.ए.)

Advisory Board

S.L. Singh E-mail vedicmri@sancharnet.in
Yeol Je Cho, E-mail : yjcho@nongae.gsnu.ac.kr
S. N. Mishra, E-mail : mishra@getafix.utr.ac.za
Som Naimpally, E-mail : sudha@accglobal.net
S. P. Singh, E-mail : spsingh@math.mun.ca
Devaki N. Talwar, E-mail : talwar@iup.edu

सम्पादक मण्डल

वीरेन्द्र अरोड़ा (गणित)
ए.के. चौपड़ा (जन्तु एवं पर्यावरण विज्ञान)
आर. डी. कौशिक (रसायन शास्त्र)
वी. कुमार (कम्प्यूटर विज्ञान)
पुरुषोत्तम कौशिक (वनस्पति एवं सूक्ष्म जीव विज्ञान)
पी. पी. पाठक (भौतिकी) प्रबन्ध सम्पादक
वीरेन्द्र अरोड़ा प्रधान सम्पादक

EDITORIAL BOARD

Virendra Arora (Mathematics)
A.K. Chopra (Zoology & Env. Science)
R. D. Kaushik (Chemistry)
V. Kumar (Computer Science)
Purushottam Kaushik (Botany & Micro.)
P. P. Pathak (Physics) Managing Editor
Virendra Arora Chief Editor

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 1-6

A NOTE ON GENERALIZATIONS OF MIDPOINT INEQUALITY

B.G. PACHPATTE*

(Received 29.01.2004)

Abstract

In this note we establish two new generalizations of the midpoint inequality involving two functions and their second order derivatives by using a fairly elementary analysis.

Mathematics Subject classification (1991) : 26 D 15, 41 A 55

Key words and phrases : Midpoint inequality, second order derivatives, integration by parts, Hölder's inequality.

INTRODUCTION

The following classical inequality is well known in the literature as the midpoint inequality:

$$\left|\int_a^b f(x)dx-(b-a)f\left(\frac{a+b}{2}\right)\right| \le \frac{\|f''\|_\infty}{24}(b-a)^3, \tag{1}$$

where the mapping $f:[a,b]\subset R\to R$ is twice differentiable on the interval (a,b) and having the second derivative bounded on (a,b), *i.e.* $\|f''\|_\infty=\sup_{x\in(a,b)}|f''(x)|<\infty$.

Recently, Dragomir, et al. [1] have established the following inequality containing an integral inequality (1).

Let $f:[a,b]\subset R\to R$ be twice differentiable mapping on (a,b) and $f'':(a,b)\to R$ is integrable on (a,b). Then

---

* 57 Shri Niketan Colony, Near Abhinay Talkies, Aurangabad 431001 (Maharashtra) India

$$\left|\int_a^b f(x)dx-(b-a)f\left(\frac{a+b}{2}\right)\right|\le\begin{cases}\dfrac{(b-a)^3}{24}\|f''\|_\infty, & \text{if } f''\in L_\infty(a,b),\\ \dfrac{(b-a)^{2+\frac{1}{p}}}{8(2p+1)^{\frac{1}{p}}}\|f''\|_q, & \text{if } f''\in L_q(a,b)\\ \qquad\text{where } \dfrac{1}{p}+\dfrac{1}{q}=1, p>1, & \\ \dfrac{(b-a)^2}{8}\|f''\|_1, & \text{if } f''\in L_1(a,b).\end{cases}\tag{2}$$

For other results related to the midpoint inequality, see Chapter XV of the book by Mitrinovic et al. [2] The main purpose of this note is to establish two new generalizations of the midpoint inequality involving two functions and their second order derivatives. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

STATEMENT OF RUESLTS

Our main results are established in the following theorems.

THEOREM 1. *Let* $f,g:[a,b]\subset R\to R$ *be twice differentiable mappings on* (a,b). *Suppose that* $f'',g'':(a,b)\to R$ *are integrable on* (a,b). *Then*

$$\left|2\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)-(b-a)\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx+g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx\right]\right|$$

$$\le\begin{cases}\dfrac{(b-a)^3}{24}\left[\|f''\|_\infty\displaystyle\int_a^b|g(x)|dx+\|g''\|_\infty\int_a^b|f(x)|dx\right], & \text{if } f'',g''\in L_\infty(a,b),\\ \dfrac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left[\|f''\|_p\displaystyle\int_a^b|g(x)|dx+\|g''\|_p\int_a^b|f(x)|dx\right], & \text{if } f'',g''\in L_p(a,b),\\ \qquad\text{where } \dfrac{1}{p}+\dfrac{1}{q}=1, q>1, & \\ \dfrac{(b-a)^2}{8}\left[\|f''\|_1\displaystyle\int_a^b|g(x)|dx+\|g''\|_1\int_a^b|f(x)|dx\right], & \text{if } f'',g''\in L_1(a,b).\end{cases}\tag{3}$$

Theorem 2. Let f, g, f'', g'' be as in Theorem 1. Then

$$\left|g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx+f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx-2(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\right|$$

$$\leq\begin{cases}\dfrac{(b-a)^3}{24}\left[\|f''\|_\infty\left|g\left(\dfrac{a+b}{2}\right)\right|+\|g''\|_\infty\left|f\left(\dfrac{a+b}{2}\right)\right|\right], \text{ if } f'',g''\in L_\infty(a,b),\\ \dfrac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left[\|f''\|_p\left|g\left(\dfrac{a+b}{2}\right)\right|+\|g''\|_p\left|f\left(\dfrac{a+b}{2}\right)\right|\right], \text{ if } f'',g''\in L_p(a,b),\\ \qquad\qquad\qquad\qquad \text{where } \dfrac{1}{p}+\dfrac{1}{q}=1, q>1,\\ \dfrac{(b-a)^2}{8}\left[\|f''\|_1\left|g\left(\dfrac{a+b}{2}\right)\right|+\|g''\|_1\left|f\left(\dfrac{a+b}{2}\right)\right|\right], \text{ if } f'',g''GL_1(a,b),\end{cases} \tag{4}$$

PROOFS OF THEOREMS 1 AND 2

Form the hypotheses and using the integration by parts formula, the following identities hold (See[1, p.26]):

$$\int_a^b f(x)dx-(b-a)f\left(\frac{a+b}{2}\right)=\int_a^b k(x)f''(x)dx, \tag{5}$$

$$\int_a^b g(x)dx-(b-a)g\left(\frac{a+b}{2}\right)=\int_a^b k(x)g''(x)dx, \tag{6}$$

where

$$k(x)=\begin{cases}\dfrac{(x-a)^2}{2} & x\in\left[a,\dfrac{a+b}{2}\right]\\ \dfrac{(b-x)^2}{2} & x\in\left[\dfrac{(a+b)}{2},b\right].\end{cases} \tag{7}$$

Multiplying both sides of (5) and (6) by $\int_a^b g(x)dx$ and $\int_a^b f(x)dx$ respectively and and adding the resulting identities we have

$$2\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)-(b-a)f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \tag{8}$$
$$-(b-a)g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx$$

$$=\left(\int_a^b g(x)dx\right)\left(\int_a^b k(x)f''(x)dx\right)+\left(\int_a^b f(x)dx\right)\left(\int_a^b k(x)g''(x)dx\right).$$

From (8) and using the properties of modulus we have

$$\left|2\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)-(b-a)\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx+g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx\right]\right|$$

$$\le\left(\int_a^b |g(x)|dx\right)\left(\int_a^b |k(x)||f''(x)|dx\right)+\left(\int_a^b |f(x)|dx\right)\left(\int_a^b |k(x)||g''(x)|dx\right). \tag{9}$$

We consider the following three cases.

(i) If $f'',g''\in L_\infty(a,b)$, then

$$\int_a^b |k(x)||f''(x)|dx\le\|f''\|_\infty\int_a^b |k(x)|dx$$

$$=\|f''\|_\infty\left[\int_a^{\frac{a+b}{2}}\frac{(x-a)^2}{2}dx+\int_{\frac{a+b}{2}}^b\frac{(b-x)^2}{2}dx\right]=\frac{(b-a)^3}{24}\|f''\|_\infty. \tag{10}$$

Similarly,

$$\int_a^b |k(x)||g''(x)|dx\le\frac{(b-a)^3}{24}\|g''\|_\infty. \tag{11}$$

Using (10) and (11) in (9) we get the first inequality in (3).

(ii) If $f'', g'' \in L_p(a,b)$ where $\frac{1}{p} + \frac{1}{q} = 1, q > 1$, then

$$\int_a^b |k(x)||f''(x)|dx \le \|f''\|_p \left(\int_a^b |k(x)|^q dx \right)^{\frac{1}{q}}$$

$$= \|f''\|_p \left(\int_a^{\frac{a+b}{2}} \left(\frac{(x-a)^2}{2} \right)^q dx + \int_{\frac{a+b}{2}}^b \left(\frac{(b-x)^2}{2} \right)^q dx \right)^{\frac{1}{q}} = \|f''\|_p \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}. \tag{12}$$

Similarly,

$$\int_a^b |k(x)||g''(x)|dx \le \|g''\|_p \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}. \tag{13}$$

Using (12) and (13) in (9), the second inequality in (3) holds.

(iii) If $f'', g'' \in L_1(a,b)$, then

$$\int_a^b |k(x)||f''(x)|dx \le \max_{x \in (a,b)} k(x) \|f''\|_1 = \frac{(b-a)^2}{8} \|f''\|_1. \tag{14}$$

Similarly,

$$\int_a^b |k(x)||g''(x)|dx \le \frac{(b-a)^2}{8} \|g''\|_1. \tag{15}$$

Using (14) and (15) in (9), we get the last inequality in (3). The proof of Theorem 1 is complete. In order to prove Theorem 2, we multiply both sides of (5) and (6) by

$g\left(\frac{a+b}{2}\right)$ and $f\left(\frac{a+b}{2}\right)$ respectively and add the resulting identities to get

$$g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx + f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx - 2(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$= g\left(\frac{a+b}{2}\right)\int_a^b k(x)f''(x)dx + f\left(\frac{a+b}{2}\right)\int_a^b k(x)g''(x)dx. \tag{16}$$

From (16) and using the properties of modulus we have

$$= \left|g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx + f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx - 2(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\right|$$

$$\leq \left|g\left(\frac{a+b}{2}\right)\right|\int_a^b |k(x)||f''(x)|dx + \left|f\left(\frac{a+b}{2}\right)\right|\int_a^b |k(x)||g''(x)|dx \tag{17}$$

The rest of the proof can be completed by following the last arguments as in the proof of Theorem 1 below the inquality (9).

In concluding, we note that, in the special case, If we take $g(x) = 1$ and hence $g''(x) = 0$ in Theorems 1 and 2, then we get the inequality (2) which in turn contains the well known midpoint inaquality given in (1).

REFERENCES

1. S.S. Dragomir, P. Cerone and A. Sofo, Some remarks on the midpoint rule in numerical integration, RGMIA research Report Collection, Vol.1, No. 2, 1998, pp 25-34.

2. D.S. Mitrinović, J.E. Pečarić and A.M. Fink, Inequalities for functions and their integrals and derivatives, Kluwer Academic Publishers, Dordrecht, 1994.

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 7-16

COMMON FIXED POINT THEOREMS FOR FOUR SELF-MAPS ON METRIC SPACES BY ALTERING DISTANCES

K.P.R. RAO* AND N. SRINIVASA RAO**

(Received 10.04.2004)

ABSTRACT

We obtain some common fixed point theorems for four self maps on metric spaces by alteration of distances using various definitions and conditions.

Key words: Fixed point, concidence point, partialy commuting, f-continuous, f-compatible, reciprocally continuous, altering distances.

2000 Mathematics subject classification : 47H10, 54H25.

INTRODUCTION

We prove some fixed point theorems for four self-maps by altering distances using various definitions and conditions.

First we give some well-known definitions.

Through out this paper, unless otherwise stated, $R(f)$ is the range of f.

DEFINITION 1. (Naidu [6]) The pair (f, g) is said to be f-continuous at z if $\{ffx_n\}$ and $\{fgx_n\}$ converge to fz whenever $\{x_n\}$ is a sequence in X such that $\{fx_n\}$ and $\{gx_n\}$ converge to z.

DEFINITION 2. (Jessy[2]) The ordered pair (f,g) is said to be weak comptaible at z if $\{gfx_n\}$ converges to fz whenever $\{x_n\}$ is a sequence in X such that $\{fx_n\}$ and $\{gx_n\}$ converge to z, and $\{fgx_n\}$ and $\{ffx_n\}$ converge to fz.

DEFINITION 3. (G.Jungck [4]) The pair (f,g) is said to be partially commuting at z or weakly compatible at z if $fz = gz$ provided there exists $w \in X$ such that $fw = gw = z$.

DEFINITION 4. (Pathak and Khan [8]) The pair (f,g) is said to be f-compatible if $d(fgx_n, ggx_n) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X such that $fx_n \to z$ and $gx_n \to z$ as $n \to \infty$.

DENINITION 5. {Jungck [3]) The pair (f,g) is said to be compatible (also called asymptotically commuting by Tiwari & Singh [15] see also [1] & [14]) if $(fgx_n, gfx_n) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X such that $fx_n \to z$ and

---

* Dept. of Applied Mathematics, Acharya N.U.P.G. Centre, Nuzvid-521201, A.P.
** Dept. Mathematics, Narasaraopeta Engineering college, Narsaraopet 522601, A.P.

$gx_n \to z$ *as* $n \to \infty$.

DEFINITION 6. (Pant [7]) The pair (f,g) is said to be reciprocally continuous on X if $\{fgx_n\}$ converges to fz and $\{gfx_n\}$ converge to gz whenever $\{x_n\}$ is a sequence in X such that $fx_n \to z$ and $gx_n \to z$ as $n \to \infty$.

DEFINITION 7. (Naidu[6]) The ordered pair (f,g) is said to be weakly f-compatible at z if either $\{gfx_n\}$ or $\{ggx_n\}$ converges to fz whenever $\{x_n\}$ is a sequence in X such that $\{fx_n\}$ and $\{gx_n\}$ converge to z, and $\{fgx_n\}$ and $\{ffx_n\}$ converge to fz.

Naidu [6] observed the following

REMARK 8.

1. If f is continuous at z, then (f,g) is f-continuous at z.
2. If (f, g) is compatible at z, then (f,g) is weak compatible at z.
3. If (f,g) is either weak compatible or f-compatible or g-compatible, then $(f,g,)$ is weakly f-compatible at z for any $z \in X$.
4. If (f,g) is weakly f-compatible at z, then it is partially commuting at z.

MAIN RESULTS

THEOREM 9. Let (X,d) be a metric space and $f,g,\ S,T\colon X \to X$ be such that

$$\begin{aligned}\psi(d(fx,gy)) \le \varphi(\max\{&\psi(d(Sx,Ty)) + \psi(d(fx,Sx) + \psi(d(gy,Ty)),\\ &\psi(d(fx,Sx)) + \psi(dSx,gy)), \psi(d(gy,Ty)) + \psi(d(fx,Ty))\})\end{aligned} \tag{9.1}$$

for all $x, y \in X$, where $\varphi\ \psi : \Re^+ \to \Re^+$, φ is monotonically incerasing on $\Re^+$, $\varphi(t+) < t$ for all $t > 0$, ψ is continuous and monotonically incereasing on $\Re^+$ and $\psi(t)=0 \Leftrightarrow t=0$.

Suppose that there is a sequence $\{x_n\}$ in X such that

$fx_{2n} = Tx_{2n+1}$ $(= y_{2n}$, say)

$gx_{2n+1} = Sx_{2n+2}$ $(= y_{2n+1}$, say) for $n = 0, 1, 2,\ldots\ldots$

and $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$. (9.2)

Then $\{y_n\}$ is a Cauchy sequence in X.

Further assume that $\{y_n\}$ converges to some $z \in X$.

Then the following statements are true.

1. If $u \in X$ and $Su = fu = u$, then $u = z$.

2. If $v \in X$ and $Tv = gv = v$, then $v = z$.
3. If $Sz = fz$, then $fz, = Sz = z$.
4. If $Tz = gz$, then $gz = Tz = z$.
5. If $Sz = z$, then $fz = z$.
6. If $Tz = z$, then $gz = z$.
7. If (f, S) is f-continuous at z and weakly f-compatible at z, then $fz = z$.
8. If (g, T) is g-continuous at z and weakly g-compatible at z, then $gz = z$.
9. If $z \in R(S)$ and (f, S) is partially commuting at z, then $fz = Sz = z$.
10. If $z \in R(T)$ and (g, T) is partially commuting at z, then $gz = Tz = z$.
11. If (f, S) is S-continuous at z and weakly S-compatiable at z, then $fz = Sz = z$.
12. If (g, T) is T-continuous at z and weakly T-compatiable at z, then $gz = Tz = z$..
13. If (f, S) is compatible at z and reciprocally continuous at z, then $fz = Sz = z$.
14. If (g, T) is compatible at z and reciprocally continuous at z, then $gz = Tz = z$.
15. If $fz = Sz = Tz$, then $fz = Sz = Tz = gz = z$
16. If $gz = Sz = Tz$, then $fz = Sz = Tz = gz = z$.
17. Maps f and S have a unique common fixed point, if one of the following statements is true.
    - (i) $fz = Sz$,
    - (ii) (f, S) is S-continuous at z and weakly S-compatible at z,
    - (iii) $z \in R(S)$ and (f, S) is partially commuting at z,
    - (iv) (f, S) is compatible at z and reciprocally continuous at z.
18. Maps g and T have a unique common fixed point, if one of the follwoing statements is true.
    - (i) $gz = Tz$,
    - (ii) (g, T) is T-continuous at z and weakly T-compatible at z,
    - (iii) $z \in R(T)$ and (g, T) is partially commuting at z.
    - (iv) (g, T) is compatible at z and reciprocally continuous at z.
19. z is the unique common fixed point of f and S, as well as g and T, if at

least one of the statements (i), (ii), (iii) and (iv) of Statement 17 and at least one of those of Statement 18 is true.

20. z is the unique common fixed point of f and S, as well as g and T, if $Sz = Tz$ and at least one of the statements (i), (ii), (iii) and (iv) of statement 17 or at least one of those of Statement 18 is true.

PROOF. Suppose that $\{y_{2n}\}$ is not Cauchy.

Then, since $\{d(y_n, y_{n+1})\}$ converges to zero, there exists $\varepsilon > 0$ and strictly increasing sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $m_k \geq n_k \geq k, \left(y_{2n_k}, y_{2n_k}\right) \geq \varepsilon$ and

$d\left(y_{2m_k}, y_{2m_k-2}\right) < \varepsilon$ for all $k = 1, 2, 3$........

we have $\varepsilon \leq d\left(y_{2n_k}, y_{2m_k}\right) \leq d\left(y_{2n_k}, y_{2m_k-2}\right) + d\left(y_{2m_k-2}, y_{2m_k-1}\right) + d\left(y_{2m_k-1}, y_{2m_k}\right)$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k}) = \varepsilon \tag{9.3}$$

Also $\lim\limits_{k\to\infty} d(y_{2n_k+1}, y_{2m_k-1}) = \varepsilon$ and $\lim\limits_{k\to\infty} d(y_{2n_k}, y_{2m_k-1}) = \varepsilon$

(from (9.2), (9.3) and triangle inequality)

From (9.1) we have

$$\psi(d(fx_{2m_k}, gx_{2n_k+1})) \leq \varphi(max\{\psi(d(Sx_{2m_k}, Tx_{2n_k+1})) + \psi(d(fx_{2m_k}, Sx_{2m_k})) + \psi(d(gx_{2n_k+1}, Tx_{2n_k+1}),$$

$$\psi(d(fx_{2m_k}, Sx_{2m_k})) + \psi(Sx_{2m_k}, gx_{2n_k+1})),$$

$$\psi(d(gx_{2n_k+1}, Tx_{2n_k+1})) + \psi(d(fx_{2m_k}, Tx_{2n_k+1}))\}),$$

That is

$$\psi(d(y_{2m_k}, y_{2n_k+1})) \leq \varphi(\max\{\psi(d(y_{2m_k-1}, y_{2n_k})) + \psi(d(y_{2m_k}, y_{2m_k-1})) +$$

$$\psi(d(y_{2n_k+1}, y_{2m_k})), \psi(d(y_{2m_k}, y_{2m_k-1})) + \psi(d(y_{2m_k-1}, y_{2n_k+1})),$$

$$\psi(d(y_{2n_k+1}, y_{2n_k})) + \psi(d(y_{2m_k}, y_{2n_k}))\}).$$

Letting $k \to \infty$, we get

$$\psi(\varepsilon) \leq \varphi(\max\{\psi(\varepsilon) + \psi(0) + \psi(0), \psi(0) + \psi(\varepsilon), \psi(0) + \psi(\varepsilon)\})$$

$= \varphi(\psi(\varepsilon))$ (since $\psi(0) = 0$).

Since $\phi(t+) < t$ for all $t > 0$, we must have $\psi(\varepsilon) = 0$.

Since $\psi(t) = 0 \Rightarrow t = 0$, we have $\varepsilon = 0$.

This is a contradiction. Hence $\{(y_{2n}\}$is a Cauchy sequence.

Since $\{d(y_n, y_{n+1})\}$ converges to zero, it follows that $\{y_n\}$ is a Cauchy.

Suppose now that $\{y_n\}$ converges to z for some $z \in X$. From (9.1), we have

$$\psi(d(fu, gx_{2n+1})) \le \varphi(\max\{\psi(d(Su, Tx_{2n+1})) + \psi(d(fu, Su)) + \psi(d(gx_{2n+1}Tx_{2n+1})),$$
$$\psi(d(fu, Su)) + \psi(d(Su, gx_{2n+1})),$$
$$\psi(d(gx_{2n+1}, Tx_{2n+1})) + \psi(d(fu, Tx_{2n+1})),$$

Letting n $\to \infty$, we get

$$\psi(d(u,z)) \le \varphi(\max\{\psi(d(u,z)) + \psi(0) + \psi(0), \psi(0) + \psi(d(u,z)), \psi(0) + \psi(d(u,z))\})$$

(since $Su = fu = u$).

Hence $d(u, z) = 0$. Therefore $u = z$. This proves (1).

(2) Follows as in (1).

(3) Suppose that $Sz = fz$.

By taking $x = z$, $y = x_{2n+1}$ in (9.1) and letting n $\to \infty$ we get.

$$\psi(d(fz,z)) \le \varphi(\max\{\psi(d(fz,z)) + \psi(0) + \psi(0), \psi(0) + \psi(d(fz,z)),$$
$$\psi(0) + \psi(d(fz,z))\}).$$

Hence $\psi(d(fz,z)) = 0$. Hence d(fz, z) = 0. Therefore $fz = z$.

(4) Follows as in (3).

(5) Suppose that $Sz = z$.

By taking $x = z$, $y = x_{2n+l}$ in (9.1) and letting n $\to \infty$, we get

$$\psi(d(fz,z)) \le \varphi(\max\{\psi(0) + \psi(d(fz,z)) + \psi(0), \psi(d(fz,z)) + \psi(0),$$
$$\psi(0) + \psi(d(fz,z))\}).$$

Hence $\psi(d(fz,z)) = 0$. Hence $d(fz,z) = 0$. Therefore $fz = z$.

(6) Follows as in (5).

(7) Suppose that (f, S) is f-continuous at z and weakly f-compatible at z.

Since $fx_{2n} \to z$, $Sx_{2n} \to z$ and (f, S) is f-continuous at z, it follows that $ffx_{2n} \to fz$ and $fSx_{2n} \to fz$. Since (f, S) is weakly f-compatible at z, it follows that either $Sfx_{2n} \to fz$ or $SSx_{2n} \to fz$.

Case (i) : Suppose that $Sfx_{2n} \to fz$. From (9.1)

$$\begin{aligned}\psi(d(ffx_{2n}, gx_{2n+1})) \le \varphi(\max\{&\psi(d(Sfx_{2n}, Tx_{2n+1})) + \psi(d(ffx_{2n}, Sfx_{2n})) + \\ &\psi(d(gx_{2n+1}, Tx_{2n+1})),\ \psi(d(ffx_{2n}, Sfx_{2n})) + \psi(d(Sfx_{2n}, gx_{2n+1})), \\ &\psi(d(gx_{2n+1}, Tx_{2n+1})) + \psi(d(ffx_{2n}, Tx_{2n+1}))\}),\end{aligned}$$

Letting $n \to \infty$, we get.

$$\begin{aligned}\psi(d(fz, z)) \le \varphi(\max\{&\psi(d(fz, z)) + \psi(0) + \psi(0), \psi(0) + \psi(d(fz, z)), \\ &\psi(0) + \psi(d(fz, z))\}).\end{aligned}$$

Hence $\psi(d(fz, z)) = 0$ and $d(fz, z) = 0$, i.e. $fz = z$.

Case (ii) : Suppose that $SSx_{2n} \to fz$.

Then by taking $x = Sx_{2n}$, $y = x_{2n-1}$ in (9.1) and letting $n \to \infty$, as in case (i) we get $fz = z$

(8) Follows as in (7).

(9) Since $z \in R(S) \exists\ w \in X$ such that $Sw = z$.

Now by taking $x = w, y = x_{2n+1}$ in (9.1) and letting $n \to \infty$ we get

$$\begin{aligned}\psi(d(fw, z)) \le \varphi(\max\{&\psi(0) + \psi(d(fw, z)) + \psi(0), \psi(d(fw, z)) + \psi(0), \\ &\psi(0) + \psi(d(fw, z))\}).\end{aligned}$$

Hence $\psi(d(fw, z)) = 0$, and $d(fw, z) = 0$ i.e. $fw = z$.

Since (f, S) is the partially commuting at z and $Sw = fw = z$, we have $fz = Sz$.

The remaining part follows from Statement (3).

(10) Follows an in (9).

(11) Suppose that (f, S) is S-continuous at z and weakly S-compatible at z.

Since $fx_{2n} \to z$ $Sx_{2n} \to z$ and (f, S) is S-continuous at z, it follows that $Sfx_{2n} \to Sz$ and

$SS_{2n} \to Sz$. Since (f,S) is weakly S-compatible at z, it follows that either $fSx_{2n} \to Sz$ or $ffx_{2n} \to Sz$.

Case (i) : Suppose that $fSx_{2n} \to Sz$

From (9.1), we have

$$\psi(d(fSx_{2n}, gx_{2n+1})) \le \varphi(\max\{\psi(d(SSx_{2n}, Tx_{2n+1})) + \psi(d(fSx_{2n}, SSx_{2n})) + \psi(d(gx_{2n+1} Tx_{2n+1})),$$
$$\psi(d(fS_{2n}, SSx_{2n})) + \psi(d(SSx_{2n}, gx_{2n+1})),\ \psi(d(gSx_{2n+1}, Tx_{2n+1})) + \psi(d(fSx_{2n} Tx_{2n+1}))\}).$$

Letting $n \to \infty$, we get.

$$\psi(d(Sz,z)) \le \varphi(\max\{\psi(d(Sz,z)) + \psi(0) + \psi(0), \psi(0) + \psi(d(Sz,z)), \psi(0)$$
$$+ \psi(d(Sz,z))\}).$$

Hence $\psi(d(Sz,z)) = 0$, and $d(Sz, z) = 0$, i.e. $Sz = z$.

Since $Sz = z$, $z \in R(S)$.

Since (f,S) is weakly S-compatible at z, it follows that (f, S) is partially commuting at z. Since $z \in R(S)$ and (f,S) is partially commuting at z, from Statement (9) it follows that $fz = Sz = z$.

Case (ii) Suppose that $ffx_{2n} \to Sz$.

Now by taking $x = fx_{2n}, y = x_{2n+1}$ in (9.1) and letting $n \to \infty$, we get $Sz = z$.

The rest of the proof follows as in Case (i).

(12) Follows as in (11).

(13) Since (f, S) is compatible, follows that $d(fSx_{2n}, Sfx_{2n}) \to 0$ as $n \to \infty$ whenever $fx_{2n} \to z$ and $Sx_{2n} \to z$ as $n \to \infty$. Since (f, S) is reciprocally continuous at z, follows that $fSx_{2n} \to fz$ and $Sfx_{2n} \to Sz$. Hence from the compatibility and reciprocal continutiy of (f,S) at z, we have $fz = Sz$.

The remaining follows from statement (3).

(14) Follows as in (13).

(15) Follows from statements (3) and (6).

(16) Follows from statements (4) and (5).

Statements (17), (18), (19) and (20) are trivial from the above proofs.

Now we state the following theorem, which is a corollary of Theorem 9.

THEOREM 10. Let (X, d) be a metric space and $f, g, S, T: X \to X$ be such that

$$\psi(d(fx,gy)) \le \varphi(\max\{\psi(d(Sx,Ty)), \psi(fx,Sx)), \psi(d(gy,Ty)),$$
$$\psi(d(fx,Ty)), \psi(d(Sx,gy))\}) \qquad (10.1)$$

for all $x, y \in X$, where $\varphi, \psi: \Re^+ \to \Re^+$, φ is monotonically increasing on $\Re^+$, $\varphi(t+) < t$ for all $t > 0$, ψ is continuous and montonically increasing on $\Re^+$ and $\psi(t) = 0 \Rightarrow t = 0$.

Suppose that there is a sequence $\{x_n\}$ in X such that

$fx_{2n} = Tx_{2n+1}$ ($= y_{2n}$, say)

$gx_{2n+1} = Sx_{2n+2}$ ($= y_{2n+1}$, say) for $n = 0, 1, 2,$............

and $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

Then $\{y_n\}$ is a Cauchy sequence in X.

Further assume that $\{y_n\}$ converges to some $z \in X$.

Then statements (1) to (20) of Theorem 9 are true here also.

REMARK 11. Theorem 10 is a generalization of Theorem 1 of Naidu & Rajendra Prasad [5], Theorem 2.1, 2.4 of Sastry et al. [13], Theorem 1 of Sastry et al. [10], Theorem 7 and 12 of Sastry et al. [11]. Also Theorem 2 of Rathore et al. [9] and Theorem 3.1 of Sastry et al. [12] are corollaries of Theorem 10.

Finally we state a theorem which is a slight variant of Theorem 9.

THEOREM 12. Let (X,d) be a metric space and $f, g, S, T: \to X$ such that

$$\psi(d(fx,gy) \le \phi(\max\{\psi(d(Sx,Ty)), \psi(d(fx,Ty)), \psi(d(Sx,gy))\} \,.$$
$$+ \psi(d(fx,Sx)) + \psi(d(gy,Ty)))$$

for all $x, y \in X$, where $\varphi, \psi: \Re^+ \to \Re^+$, φ is monotonically increasing on $\Re^+$, $\varphi(t+) < t$ for all $t > 0$, ψ is continuous and montonically increasing on $\Re^+$ and $\psi(t) = 0 \Rightarrow t = 0$.

Suppose that there is s sequence $\{x_n\}$ in X such that

$fx_{2n} = Tx_{2n+1}$ ($= y_{2n}$, say)

$gx_{2n+1} = Sx_{2n+2}$ ($= y_{2n+1}$, say) for $n = 0, 1, 2,$............

and d $(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

Then $\{y_n\}$ is a Cauchy sequence in X.

Further, assume that $\{y_n\}$ converges to some $z \in X$.

Then the following statements are true.

1. If $u \in X$ and $Su = fu, = u$ then $u = z$.
2. If $v \in X$ and $Tv = gv = v$, then $v = z$.
3. If $Sz = fz$, then $fz = Sz = z$.
4. If $Tz = gz$, then $gz = Tz = z$.
5. If (f, S) is f-continuous at z and weakly f-compatible at z, then $fz = z$.
6. If (g, T) is g-continuous at z and weakly g-compatibleat z, then $gz = z$.
7. If (f, S) is S-continuous at z and weakly S-compatible a z, then $Sz = z$.
8. If (g, T) is T.continuous at z and weakly T-compatile at z, then $Tz = z$.
9. If (f, S) is compatible at z and reciprocally continuous at z, then $fz = Sz = z$.
10. If (g, T) is compatible a z and reciprocally continuous at z, then $gz = Tz = z$.
11. Maps f and S have unique common fixed point if one of the following statements is true.

    (i) $fz = Sz$

    (ii) (f, S) is comatible at z and reciprocally continuous at z.
12. Maps g and T have unique common fixed point if one of the following statemts is true.

    (i) $gz = Tz$

    (ii) (g, T) is comatible at z and reciprocally continuous at z.
13. Z is unique common fixed point of f and S, as well as g and T, if at least one of the following statement (i) and (ii) of statement (11) and at least on of those of statement (12) are true.

ACKNOLOWLEDGEMENTS: The authors express their heratfelt thanks to Prof. S.V.R. Naidu for his valuable suggestions in the preparation of this paper.

REFERENCES

1. U.C. Gairola, S.L. Singh and S.H.M. Whitfield : fixd point theorems on product of compact metric space, Cemoust. Math. 28 (1995) 541-548.
2. A. Jessy : Studies in fixed points and quasi-guages, Ph.D, Thesis, I.I.T. Madras, 1991.
3. G. Jungck : Compatible mappings and common fixed points, Internat. J. Math. & Math. Sci. 9 (4) (1986) 771-779.
4. G. Jungck : Common fixed points for non-continuous non-self maps on nonmetric spaces, Far East J. Math Sci 4 (2) (1996) 199-215.
5. S.V.R. Naidu and J. Rajendraprasad : Common fixed points for four self-maps on a metric space, Indian J. Pure Appl. Math. 16 (10) (1985) 1089-1103.
6. S.V.R. Naidu : Fixed point theorems for four self-maps on a metric space by altering distance, J. Indian Math. Soc. (to appear).
7. R.P. Pant : A common fixed point theorem under a new condition, Indian J. Pure Appl. Math. 30 (2) (1999) 147-152.
8. H.K. Pathak and M.S. Khan : A comparison of various types of compatible maps and common fixed points, Indian J. Pure Appl. Math. 28 (4) (1997) 477-485.
9. M.S. Rathore, Mamata Singh, Satira Rathore and Naval Singh : Concepts of the set $E\alpha$ and common fixed points, Bull. Cal. Math. Soc. 94 (2002) 259-270.
10. K.P.R. Sastry, S.V.R. Naidu, I.H.N. Rao and K.P.R. Rao : Common fixed points for asymptotically regular mapping, Indian. J. Pure Appl. Math. 15 (8) (1984) 849-854.
11. K.P.R. Sastry, S.V.R. Naidu, I.H.N. Rao and K.P.R. Rao : Common fixed points under asymptotic regularity condition, Nep. Math. Sc. Rep. 9 (12) (1984) 63-71.
12. K.P.R. Sastry, G.V.R. Babu and D. Narayana Rao : Fixed points theorems in complete metric space by using a continuous control function, Bull. Cal. Math. Soc. 91 (6) (1999) 493-502.
13. K.P.R. Sastry, S.V.R. Naidu, G.V.R. Babu and G.A. Naidu : Generalisation of common fixed point theorems for weakly commuting maps by altering distances, Tamkang. J. Math. 31 (3) (2000) 243-250.
14. S.L. Singh and S.N. Mishra : Remarks on recent fixed points theeems and applications to integral equations, Demost. Math. 34 (4) (2000) 847-857.
15. B.M.L. Tivari and S.L. Singh : a note on recent generalizations of Jungck contration principle, J. UPGC Acad. Soc. 3 (1986) 13-18.

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 17-20

HARMFUL BIOEFFECTS OF HIGH FREQUENCY ELECTROMAGNETIC FIELDS

VIJAI KUMAR*, R.P. VATS* AND P.P. PATHAK**

(Received 05.06.2004)

ABSTRACT

The influence of electromagnetic field (EMF) of different frequencies at different distances from the transmission tower is discussed Human body is a dielectric and consists of complex matrices of electric and magnetic fields. When the body is exposed to high frequency radiation, the electric field is induced inside the body. It is found that the current density at the surface of skin and bone increase with frequency of radiation being exposed High frequency radation also produces the dangerous deseases as eukemia and cancer risk in childhood.

Key works: Current density, electromagnetic field, conductivity of skin and bone

INTRODUCTION

Due to excessive modernization human beings are always surrounded by electromagnetic environment. Broadcast and communication transmitters as well as home appliances like induction heaters, VDU and TV sets and medical equipments emit EMFs of different frequencies and intensities. Though, EMFs of low frequencies are not so injurious but all types of high frequency EMFs affect the human body in many ways causing many diseases such as leukemia, brain tumor, and cancer risk in childhood [5,6,12]. These radiations affect the tissues, cells and skin of human body by induction of the electric and magnetic fields inside the body on exposure to the radiation. It seems well suited to predict changes in thermo- regulatory responses that may result from deposition of EM energy into the body and has provided the basis for several such analysis [4, 14, 16]. The EM radiation influences themoregulatory behaviour like other sources of heat Hardy and Guieu [7] has summarized its themophysiological effects of human body. Similarly, work has been done on physical character of human body in relation to heat loss [3]. If only one part of body (such as head or trunk) is exposed to EM radiation, the change in selected ambient temperature range is governed by an integrated energy deposition over the whole body and not by energy deposited in some specific locus as the brain. Indeed the hypothalamus appears to play no more vital role in behavioural thermoregulation during EM exposure than what the other thermo senstive regions of the body do [1]. Therefore interaction may be expected between the body and radiation resulting into many types of biological effects. Efforts have been made to decribe this type of interaction by Polk [11] and Mason et al. [9]. In the present paper current density in

---

*Department of Physics, M.S. College, Saharanpur (U.P.)
**Deptartment of Physics, Gurukul Kangri University, Haridwar

human skin and bone due to electromagnetic environment have been studied.

MECHANISM

The EMF produces two type of effects that control the dielectric behaviour. On is the oscillation of free charges or ions and the other is the rotation of dipole molecules at the frequencies of the applied EM energy. The first gives rise to conduction currents with an associated energy loss due to electrical resistance of the medium with an associated dielectric loss due to viscosity. Here we study the current density in the skin and human body induced due to the radiation of 10 kW transmission towers. Propagation of these rediations in the atmosphere means the propagation of electric and magnetic fields around the tower. These fields decrease as inversely proportional to distance from the tower. Simultaneously the conductivity of skin varies with frequency of radiation. If σ is the conductivity and E is the exposed electric fields, the surface current density in the skin is given by

$$\mathbf{J} = \sigma\, \mathbf{E}$$

The magnitude of induced electric field at a distance r from the antenna tower emitting at 10 kW is given by Pathak et al. [10]

$$E = 774.6/r$$

RESULTS AND DISCUSSION

The conductivity of skin and bone are increased with radiation frequency of exposure to body reported by Johnson and Guy [8]. Using these data calculated values of current density induced on the skin of human body at distances 100 m, 300 m, 500 m and 1 km from the tower are shown in Table 1. Table 2 shows calculated values of current density induced on bones of human body at various distance of 100 m. 300 m, 500 m and 1 km from the tower. Current flow or current density may affect the flow of ions. Watchtel [15] observed shift in neuron firing rate response to injected transmembrane. He also injected low frequency current into the seawater surrounding the cell preparation through external electrodes and observed some variation in effects. Shwan [13] proposes that current density in the tissues is the most useful parameter to the hazards. But the exact mechanism is to be worked out as to how EMF produces the heat inside the biological material, which also breaks down the protective mechanisms of heat control inside the body [2].

The values of current densities on the skin and bone of human beings are also increased with frequency of radiation as shown in Tables 1 and 2, which affect the human body in various ways. Thus the presence of transmission tower is harmful for the public health and therefore should be installed far away from populated areas.

Table 1: Variation of conductivity and current density on the skin of human beings with high frequency ratiation.

Frequency of radiation (MHz)	Conductivity (mho)	Current density at different distances from the tower			
		100m	300m	500m	1 Km.
10	0.625	0.0893	0.0298	0.0178	0.00893
27.12	0.662	0.1337	0.0445	0.0267	0.0133
40.68	0.693	0.1621	0.54	0.0323	0.1614
100	0.889	0.28	0.0934	0.056	0.028
200	1.28	0.5081	0.1694	0.1063	0.0506
300	1.37	0.5684	0.1894	0.1135	0.0567
433	1.43	0.6034	0.2031	0.12	0.06034
750	1.54	0.6622	0.02208	0.1349	0.662
915	1.6	0.7008	0.2337	0.1401	0.07008
1500	1.77	0.8053	0.2686	0.1612	0.805
2450	2.21	1.047	0.3491	0.2095	0.1045
3000	2.26	1.0938	0.3645	0.2187	0.1091
5000	3.92	1.979	0.6597	0.3959	0.1975
5900	4.73	2.421	0.8083	0.4848	0.2421
8000	7.65	4.2304	1.4106	0.846	0.4222
10000	10.3	5.7062	1.9034	1.1422	0.5706

Table 2 : Variation of conductivity and current density in the bones of human beings with high frequency ratiation.

Frequency of radiation (MHz)	Conductivity (mho)	Current density at different distances from the tower			
		100m	300m	500m	1 Km.
27.12	27.05	28.564	9.5216	5.7129	2.853
40.68	32.07	45.747	15.238	9.1527	4.5714
100	47.5	116.802	38.931	23.355	11.67
200	60	175.38	58.458	35.37	17.52
300	69.3	209.078	69.708	41.787	20.893
433	77.95	238.293	79.446	47.658	23.501
750	93.9	287.052	95.702	57.41	28.31
915	101.3	309.674	103.244	61.934	30.541
1500	120.9	369.591	123.221	73.918	36.451
2450	154.7	479.26	159.774	95.821	47.895
3000	172	532.856	177.641	106.571	53.251
5000	235.5	729.579	243.224	145.915	72.91
5900	262	863.552	287.859	172.684	85.276
8000	343	1189.52	396.542	237.87	118.849
10000	436.5	1560.48	520.133	312.053	155.917

REFERENCES

1. E.R. Adair : Mirowave challenges to the themoregulatory system, in Electromagnetic Waves and Neurobehavioral Function, M.E.O'Connor and R.H. Lovely, Eds., Alan R. Liss, New York 1988,179.
2. E.R. Adair : Themoregulation in the presence of microwave fields, in Handbook of Biological Effects of Electromagnetic Fields, C. Polk and E. Postow , Eds., CRC Press Inc. Boca Raton, 1996.
3. I.G. Berglund : Characterizing the thermal environment is microwave and themoregulation, E.R. Adair, Ed, Academic Press, New York, 1983. 15.
4. A.F. Emery, R.E. Short, A.W. Guy, K.K. Kraning, and J.C.Lin : The numerical thermal simulation of the human body when undergoing exercise or nonionizing electromagnetic irradiation, Trans. Am. Soc. Mech. Eng. 284, 1976.
5. M. Feychting, and A. Ahlbom : Magnetic fields and cancer in people residing near Swedish high power lines. Am.J. Eypedmiol. 138 (1993) 467.
6. M. Feychting and A. Ahlbom : Magnetic fields and cancer in people residing near Swedish high power lines. Institutet for miljomedicin, Karolinska intitutet, 1992.
7. J.D. Hardy and J.D. Guieu : Interaction activity of preoptic units II. Hypothetical network, J. Physiol.(Paris) 63 (1991) 264.
8. C.C. Johnson, and A.W. Guy : Non-ionising electromagentic effect in biological materials and systems. Proc. IEEE, 18 (June 1972) 692-718.
9. P.A. Mason, W.D. Hurt, T.J. Walters. , J.A. D Andera, P. Gajsek, K.L. Ryam, K.I. Nelson and J.M. Zinax : IEEE Trans Microwave Theory and Tech. 48 (11) (2000) 2050-2058.
10. P.P. Pathak , V. Kumar, and R.P. Vats : Harmful electromagnetic environment near transmission tower, Indian J. Radio & Space Phys. 32 (2003) 238-241.
11. Polk C : Introduction, in Handbook of Biological Effects of Electromagnetic fields, C. Polk and E. Postow (Eds.) CRC press Inc. Boca Raton, 1996.
12. D.A. Savitz, E.M. John and R.C. Kelckner : Magnetic field exposure from electric appliances and childhood cancer, Am. J. Epidemiol. 131 (1990) 763.
13. H.P. Schwanv : Interaction of microwave and radio frequency radiation with biological systems, IEEE Trans. Microwave Theory Tech, Special Issue on Biological Effects of Microwaves) MTT- 19 (1971) 146-152.
14. R.J. Spiegel, D.M. Deffenebuag, and J.E. Mann : Athermal model of the human body exposed to an electromagnetic field, Bioelectromagnetics, 1 (1980). 253.
15. H. Watchel : Firing pattern changes and transmembrance currents produced by low frequency fields in pacemaker neurons, in Proc 18th Annu Hansford Life Sci. Symp. Technical information center, U.S. Department of Energy, Richland, Washington 1978. 132.
16. W.L. Way, H. Kritkos, and H.P. Schwan : Thermoregulatory physiologic responses in the human body exposed to microwave radiation, Bioelectromagnetics, 2 (1981) 341.

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 21-34

METHODS FOR SOLVING CUBIC EQUATIONS

ANITA SHARMA* AND V.K. SHARMA*

(Received 06.05.2004 and in Revised form 17.07.2004)

ABSTRACT

In this paper different methods for solving cubic equation are given. The contribution of Indian and Foreign mathematicians in the development for solving cubic equation are also discussed. Specially Omar Khayyam's geometrical method for solving particular type of cubic equation and Vieta's method for solving a cubic equation by converting it into quadratic are also explained.

Key Words : Cubic, Quadratic, Real, Imaginary

Classification No. : 01A32

INTRODUCTION

An equation of the form $ax^3+bx^2+cx+d=0$ where a, b, c and d are real numbers and $a \neq 0$ is called real cubic equation.

The oldest cubic equation which is of the form $x^3= k$ is due to Mehaxchumus (c. 350 B.C.) [7].

The next cubic equation is due to Archimedes (c.200 B. C.) [7].

$$x^3+c^2b=cx^2$$

Eutocius (c. 560 A.D.) solved the cubic equation of Archimedes by finding the intersection of two conics [7].

$$x^2 = (a^2/c^2)y \text{ (parabola)}$$

$$y(c-x) = bc \text{ (hyperbola)}$$

Next development of cubic equation was due to Arabs and Persians. Almahani (c. 860 A.D.) solved the cubic equation of Archimedes but he contributed nothing new. In pursuing the solution of cubic equation , Omar Khayyam (c. 1100 A.D.) blended solid geometry and geometry of conics. He gave geometrical and graphical solutions [3, 4, 7, 8], which are as follows :

---

*Department of Mathematics and Statistics, Gurukul Kangri Vishwavidyalaya, Haridwar.

(a) Geometrical solution

let $x^3+a^2x=b$ — (1) be a cubic equation, which is the intersection of two curves

(i) $x^2=ay$ (Parabola)

(ii) $x^2+y^2=(b/a^2)\,x$ (Circle)

Now there are four possibilities :

(1) when a and b are positive particularly we illustrate a=1 and b=2

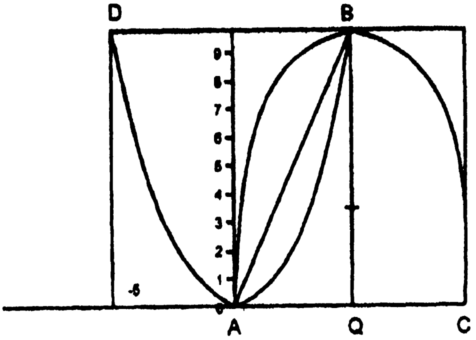


Fig. : 1

AQ=1 is the required solution of eqn (1).

(2) a= -1 (negative) and b=2 (positive), we must use the semicircle in quadrant 4

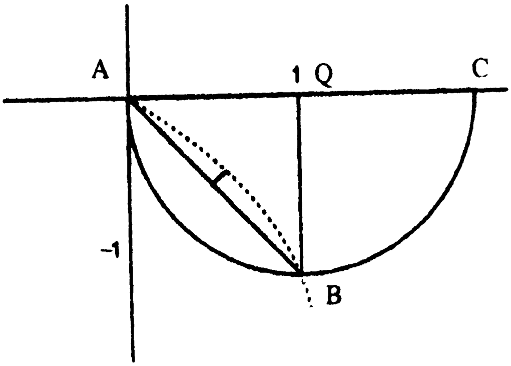


Fig. : 2

AQ =1 is the required solution of eqn (1).

(3) a= -1 (negative) and b= -2 (negative) because diameter b/a^2 can not be negative

so if b is negative, semicircle shifts to the left of the vertical axis.

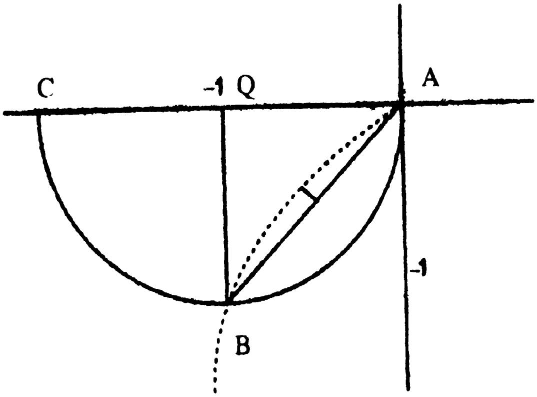


Fig. : 3

AQ=-1 is the solution of eqn(1).

(4) a =1 (positive) and b= -2 (negative)

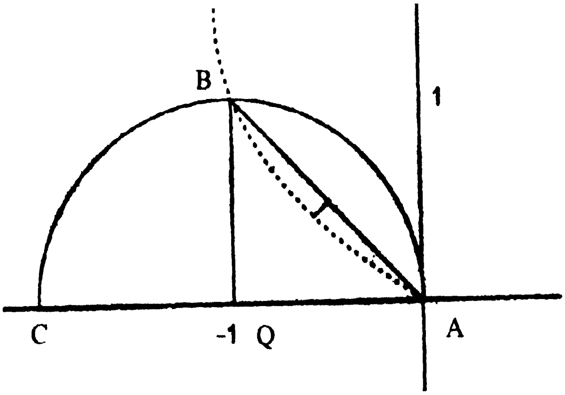


Fig. : 4

AQ=-1 is the solution of eqn(1).

(b) Graphical solution

Graphical solution of cubic equation is known from ancient time. It is also found in Omar Khayyam's work [1,4,6,10]. It is as follows :

Example: let $x^3-5x^2+4x = 0$ be a cubic equation

$$y = x^3 - 5x^2 + 4x$$

for different values of x, different values of y can be obtained

x	-2	-1	0	1	2	3	4	5
y	-36	-10	0	0	-4	-6	0	20

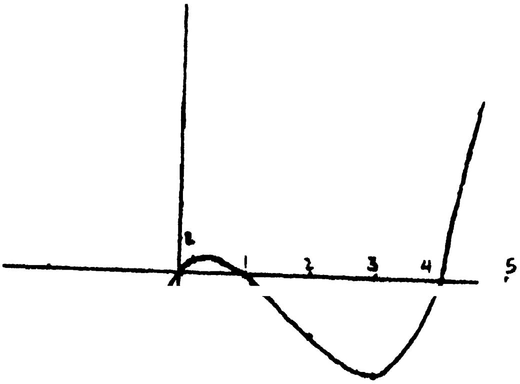


Fig. : 5

Graph cuts the x axis at three points 0, 1, 4, So 0, 1 and 4 are the roots of cubic equation.

Mahāvirāchārya (c 860 A.D.) considered equation of the following form by which only particular type of cubic could be solved [3,7]. It is as follows:

$$Ax^n=q$$

Bhāskarāchāryā (c. 1150 A.D.) wrote a problem of cubic equation in his Bijgaṇita [3, 5, 7] which is as follows:

राशिद्वादशनिघ्नो राशिघनाढ्यश्च कः समो यः स्यात्।
राशिकृतिः षड्गुणिताः पञ्चत्रिंशद्युता विद्वन् ॥ [5, P. 233]

This means

What is the number which multiplied by 12 and increased by the cube of number is equal to six times the square of number added with 35.

Assuming the number to be x, then we should have

$$x^3+12x = 6x^2+35$$

$$x^3+12x-8 = 6x^2+35-8$$

$$(x-2)^3 = 3^3$$

x= 5 is the root of equation.

European mathematician Fibonacci (c. 1225 A.D.) studied cubic equation in his work Felos and Liber Quadratorum [2, 7, 9]. For the problem $x^3+2x^2+10x=20$, he proved that x can not be irrational such as $\sqrt{a+\sqrt{b}}$.

Example: Consider the following cubic equation.

$$x^3+2x^2+10x = 20 \text{ ---------------------------------------(2)}$$

since $2 < x < 1$

So, x must have a value of the form a/b. From eqn.(2)

$$\frac{a^3}{b^3}+\frac{2a^2}{b^2}+\frac{10a}{b}=20$$

which can not be possible. From eqn.(2)

$$x = (20-2x^2)/(10+x^2)$$

put x= $\sqrt{t}$

$$\sqrt{t} =(20-2t)/(10+t)$$

which is impossible.

He also gave an approximate solution in sexagesimal form

$$x = 1, 22, 7, 42, 33, 4, 40$$

$$x = 1+\frac{22}{60}+\frac{7}{60^2}+\frac{42}{60^3}+\frac{33}{60^4}+\frac{4}{60^5}+\frac{40}{60^6}$$

In decimal notation

$$x \sim 11368808106$$

Scipio del Ferro (1465 - 1526 A.D.) and Cardan published the general method for solving cubic equation by converting it into quadratic [1, 2, 9]. Let the cubic equation be

$$c_1x^3+c_2x^2+c_3x+c_4 = 0 \quad \text{------------------} (3)$$

or $$x^3+px^2+qx+r = 0 \quad \text{------------------} (4)$$

where $p = c_2/c_1$, $q = c_3/c_1$ $r = c_4/c_1$

put $x = y - p/3$ in eqn. (4)

we get $$y^3 + Ay + B = 0 \quad \text{------------------} (5)$$

where $$A = \frac{1}{3}(3q - p^2)$$

and $$B = \frac{1}{27}(2p^3-9pq+27r)$$

$$y = m^{1/3} + n^{1/3}$$ be the solution of eqn. (5)

$$y^3 - 3m^{1/3}n^{1/3} - (m+n) = 0 \text{------------------} (6)$$

comparing eqn. (5) and eqn. (6)

$$m^{1/3}n^{1/3} = -A/3$$

$$m + n = -B$$

If m and n are the roots of quadratic equation then equation is-

$$t^2+Bt-A^3/27 = 0$$

$$t = \frac{-B \pm \sqrt{B^2 + 4A^3/27}}{2} \quad \text{------------------} (7)$$

taking cube root of m and n, we get value of y

$$y = \left[\frac{-B}{2}+\left(\frac{B^2}{4}+\frac{A^3}{27}\right)^{1/2}\right]^{1/3} - \left[\frac{B}{2}+\left(\frac{B^2}{4}+\frac{A^3}{27}\right)^{1/2}\right]^{1/3} \text{------------}(8)$$

three conditions are possible for the roots of equation

1) $B^2+4A^3/27 < 0$, all roots are real but eqn (8) gives them in an imaginary form, thus Cardan's solution is not convenient.

(2) $B^2+4A^3/27 > 0$, one root is real, other two are pair of complex conjugate.

(3) $B^2+4A^3/27 = 0$, cubic consists two equal roots.

Cardan also proved this formula geometrically. The proof of formula is given in the end of paper.

In 1545 A.D. Cardan published the solution of following equation [9]

$$x^3+mx=n \quad \text{---}(9)$$

and $$(a-b)^3+3ab(a-b) = a^3-b^3 \quad \text{---}(10)$$

comparing eqn. (9) and eqn. (10), we get

$$3ab = m$$

$$a^3-b^3 = n$$

then (a-b) is one of the solution of eqn. (9) .

Vieta (b. 1540 A.D.) and Harriat (b. 1560 A.D.) published following method for solving cubic equation (removing x^2 form) [2, 7]

$$x^3 + 3b^2x = 2c^3 \quad \text{-------------------------------------}(11)$$

put $x = (e^2-b^2)/e$ in eqn. (11)

we get $$e^6-2c^3e^3=b^6 \quad \text{---}(12)$$

this equation is quadratic in e^3 and then $x = e-b^2/e$, e, and b^2/e are the roots of expression (11).

Vieta (b. 1540 A.D.) and Girard (b. 1629 A.D.) compared the cubic eqn. (9)

with trigonometrical identity [4, 7]

$\mathrm{Cos}^3\theta - 3/4\ \mathrm{Cos}\,\theta = 1/4\ \mathrm{Cos}3\theta$ ------------------------------(13)

Example: Find depth z to which a spherical solid ball made of cork (sp. qr. 1/4) will sink into water

Solution: let radius of ball be a. By Archimedes principle

$z^3 - 3az^2 + a^3 = 0$--(14)

put $z = nx + a$ in eqn. (14)

we get $n^3x^3 - 3a^2nx - a^3 = 0$ --(15)

for $x = \mathrm{Cos}\,\theta$ eqn. (13) is identical with eqn. (15)

$n = 2a$ and $\cos 3\theta = 1/2$

for three distinct values of θ 20^0, 100^0, 140^0 three values of z can be obtained

Required depth $z = a + 2a\ \mathrm{Cos}\ 100^0$

This method is suitable when all roots are real, for imaginary roots identity involving hyperbolic function can be used.

Bomballi (c. 1572 A.D.) introduced complex numbers by which the solution of irreducible case could be found by Cardan rule. It was also given by Kastner (b. 1745 A.D.) and Chariout (b. 1746 A.D.) [2, 7]. Following is an example :

Example: let $x^3 - 63x - 162 = 0$ -------------------------- (16)

be a cubic eqn. to solve

from eqn. (8)

$$x = [(81 + 30\sqrt{-3})]^{1/3} - [(81 - 30\sqrt{-3})]^{1/3}$$

taking cube root $x = -6$ is one of the root of given cubic equation. Dividing the eqn. (16) by $(x+6)$, we get

$x^2 - 6x - 27 = 0$

where roots are 9 and -3

It means three roots of cubic equation are -6, 9, -3.

In 1668, Hudde gave another method for solving cubic equation [7]

let $x^3 = qx+r$ --------------------------------------(17)

put $x = y+z$ in eqn.(17)

$y^3+z^3 + 3yzx = qx+r$ --(18)

comparing coefficients of like terms

we get $y^3+z^3 = r$

$3yz = q$

solving for y and z we get value of x.

Newton (c.1669A.D.) suggested that a particular root of a equation could be found by substituting a known approximate root in the given equation [4, 7]. Following is an example :

Example:- let the cubic equation be

$y^3 - 2y - 5 = 0$--------------------------------------(19)

for $y = 2$

$y^3 -2y -5 = -1$

for $y = 3$

$y^3 -2y -5 = 16$

$2 < y < 3$

let $y = 2 + p$ - --------------------------------(20)

from eqn. (19)

$(2 + p)^3 - 2p\,(2+p) - 5 = 0$

$p^3 + 6p^2 + 10p - 1 = 0$--------------------------------------(21)

we get $p = 0.1$ approximately

put $p = 0.1+q$ --(22)

we have

$q^3 + 6.3\,q^2 + 11.23q + 0.061 = 0$ -----------------------------------(23)

$q = -0.0054$ approximately

put $q = -0.0054 + r$ --(24)

from eqn. (23) (omitting r^3 term)

$6.3r^2 + 11.1619r + .000541708 = 0$ ---------------------------(25)

$r = -0.00004854$ approximately

$r = -0.00004854 + s$----------------------------------(26)

from eqn. (25) (omitting s^2 term because s^2 is very small real number)

$177076885\,s = 1.2584$

$s = 0.00000000\ 7106$ approximately

$s = 0.00000000\ 7106 + t$

being a very small real number we will not consider the value of t

Put the value of s in eqn.(26)

we get

$r = -\ 0.000048532894$

from eqn. (24)

$q = -\ 0.005448532894$

from eqn. (22)

$p = 0.09451467106$

and finally we have

$y = 2.09451467106$ (approximately)

The process terminates if the root is commensurable and it may be carried to any required number of decimal places if it is incommensurable.

De Moivre's (1667-1754 A.D.) method for solving cubic equation was based on the identity [1,4]

$$(\mathrm{Cos}\,\theta + i\,\mathrm{Sin}\,\theta)^n = \mathrm{Cos}\,n\theta + i\mathrm{Sin}\,n\theta$$

let the cubic equation be $ax^3+bx^2+cx+d=0$ and can be written as

$$(x+b/3a)^3 = (b^3/27)a^3 - d/a \text{ ------------------------------------(27)}$$

if $b^2 = 3ac$

Example : let the cubic equation be

$z^3+6z^2-12z-12 = 0$, from eqn. (27) we get

$$(z-2)^3 = 4$$

$$(z-2)^3 = 4(\cos 2r\pi + \sin 2r\pi)$$

$$(z-2) = 4^{1/3}(\cos 2r\pi + \sin 2r\pi)^{1/3}$$

for r = 0, 1,2 we get three roots. De Moivre's method can be applied when R.H.S. of eqn. (22) is not real. Pure algebric method fails to find the cube root of general complex number in explict form. When $b^2= 3ac$ is not satisfied, this method can be used only particular cases.

Lagrange (1736-1813 A.D.) introduced a method of expressing the root of numerical equation in the form of continued fraction [1].

Example: $x^3-2x-5 = 0$ ------------------------------(28)

root lie between 2 and 3

put $x = 2+1/y$ in eqn. (28)

transformed eqn. is

$$y^3-10y^2-6y-1=0 \text{ ---------------------------------(29)}$$

root lie between 10 and 11

put $y = 10+1/z$ in eqn. (29), we get

$$61z^3-94z^2-20z-1 =0 \text{ ------------------------------(30)}$$

$$1< z < 2$$

put $z = 1+1/u$ in eqn. (30)

$$54u^3+25u^2-89u-61=0$$

$$1 < u < 2 \text{ and so on-----}$$

expression for the root

$$x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + }}} \text{—}$$

Geometrical proof of formula

Cardan's geometrical proof of his formula given by eqn. (8) is as follows :

Let the cubic equation be $x^3+mx=n$------------------------(31)

where m and n are positive because diamensions of cube be positive. Assuming a cube Z with side t. Where $t = (t-u) + u$

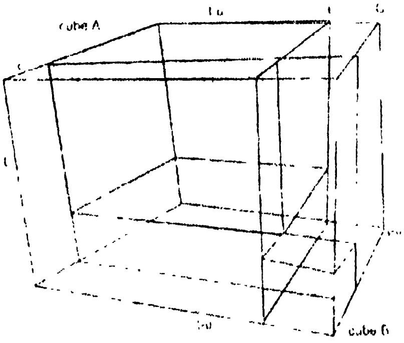


Fig. : 6

This cube of side t contains two cube A and B having sides (t-u) and u and four cuboids.

vol. of cube Z = vol. of cube A+ vol of cube B+vol. of four cuboids

$$t^3 = (t-u)^3 + u^3 + 2tu(t-u) + (t-u)u^2 + u(t-u)^2$$

$$t^3-u^3 = (t-u)^3 + (t-u)\,3tu \text{ ----------------(32)}$$

let $x = t-u$ then

$$t^3-u^3 = x^3+ 3tux \text{ -------------------------(33)}$$

comparing eqn. (31) and eqn. (33)

$$m= 3tu \text{ ---------------------------------(34)}$$

$$n = t^3- u^3 \text{ ------------------------------(35)}$$

from eqn. (34) and eqn. (35), we get

it means $t^6-nt^3-m^3/27= 0$ ----------------------------------(36)

$$t = [+n/2 + (n^2/4 + m^3/27)^{1/2}]^{1/3}$$

from eqn. (35)

$$u = [-n/2 + (n^2/4 + m^3/27)^{1/2}]^{1/3}$$

and finally $x= t-u$ is the solution.

CONCLUSION

Omar Khayyam's, Bhāskarācharyā's, and Eutocius's methods are suitable only for solving particular type of equations. Methods given by Ferro, Cardan, Bomballi, Vieta, Girard and Newton are general methods. Numerical methods based on convergence of sequences are now used for solving cubics.

REFERENCES

1. W. S. Burnside, A.W. Panton, Theory of Equations, S. Chand and Co. Delhi 1957.
2. J.N. Crossley, The Emergence of Number, World Scientific Publishing Co. Pte. Ltd. Singapore 1987.
3. B. Dutta and A.N. Singh, History of Hindu Mathematics, Asia Publishing House, Bombay, 1962.
4. R.C. Gupta, Many Methods of Solving a Cubic JOBIC, 1961-62, 54-63.
5. पं० देवचन्द्र झा "बीजगणितम्" कृष्णदास अकादमी चौखम्बा प्रेस वाराणसी। १९८३
6. Lancolet Hogben, Mathematics for the Million, George Allen and Inwin Ltd. Museum Street, London 1957.

7. D.E. Smith, History of Mathematics Vol. II, Dover Publications New York 1959.

8. G.S. Smirnova, Geometrical Solutions of Cubic Equation in Raffaele Bomballi's Algebra Istor, Metodol Estestv, Nauk 36, 1989, 123-129.

9. V.S. Vardhrajan, Algebra in Ancient and Modern Times, Hindustan book Agency, New Delhi 1997.

10. P.D. Yardley, Graphical Solution of Cubic Equation developed from the work of Omar Khayyam Bull Inst Math Appl 26 (5-6) 1990 122-125.

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 35-46

CONSTRUCTION OF SOME IMPORTANT VEDIS FROM ŚULBA SŪTRA

YOGITA BANA* AND V.K. SHARMA*

(Received 06.05-2004, Revised 19-07-2004.

ABSTRACT

Construction of some vedis from Sulba Sūtra are given.

Key words and phrases: Prācī, Sacifice, Oblong, Citi, Isosceles trapenzium, Altar, Kāmya Agni, Nitya Agni,

Classification Number: 01A32

INTRODUCTION

The science of Mathematics which is said to be the " queen of all sciences" or " the peak of all sciences" has originated in so hoary manner that it is impossiable to trace its origin in any of the written records of the human civilization. Mathematics was known in ancient India as Ganir-veda or the Veda (i.e. Knowledge) of cacluationls for detail description [15].

Only seven Sulba Sūtra are available at present. These are : 1 Baudhāyana, 2. Āpastamba, 3. Vādhula, 4. Maitrāyana, 5. Manava, 6. Vārāha, 7. Kātyāyana. First six belong to Traittriya Samhitā of Krsna Yajurveda. Only the sūtra of Kātyāyana belongs to S̄ukla Yajur-veda. Out of these Baudhāyana, Āpastamba and Kātyāyana deal with the mathematical portion-namely Geometry, in detial Manava gives a very scanty description of the geometry. It deals only with the Square and Circle. The Baudhāyana, Āpastamba and Manāva make use of these rules in the construction of various citis. But description gives only their dimensions accordingly to their shapes.

The most important points observed in all of them are:-

1. The shape of the citis or the altars of the sacred fires may differ but their areas will remain the same.
2. In the construction of citis, there will be five layers and each layer will consist Department of Mathematics & Statistics, Gurukula Kangri Vishvavidyalaya, Haridwar.of 200 bricks.
3. In the case of the three sacred fires only Garhapatya will have five layers and each layer will consist of 21 bricks [12].

---

* Department of Mathematics & Statistics, Gurukul Kangri Vishvavidyalaya, Haridwar.

GLIMPSES OF FIRE-ALTARS

There is fundamental difference between the two types of ritual. The grahya ritual is performed by the householder and his wife in the Gārhapatya fire only. In this ritual, as mentioned in Rv. X 8.5.27 "In this thy (husband's family) affection may increase with offspring, be watehful over the Garhapatya Agni in this house." A priest is seldom required. The srauta ritual is performed by the sacrificer and his wife in more than one fire viz. Gārhapatya, Āhavaniya and Daksina with important role of priest. However there are various views in concern of different agnis on the basis of seasonal and periodical sacrifices as well as Nitya and Kamya.

Kāmya agnis were having five layers, each made of fixed number of bricks. The Nitya fire has 21 brick and kamya fire has 200 bricks in each layer [10].

Name	Horizontal Section/Shape	Sūtra reference
Nitya Agni		
Āhavaniya	Square	Baudhāyana Sulba sūtra (BSS 7.4-7.7)
Gārhaptya	Circle	BSS 7.8
Daksinagni	Semi Circle	[13]

Kamya Agni

Optional fire altars which are built for the fulfilment of desire i.e. Kamyagni Taittariya-Samhita 5.4.11.1-3. These are:- (Each optional fire has an area 7½ square purusa) (see 13).

Name	Horizontal Section/Shape	Sutra reference
Kamya Agni		
Caturasrasyenacit	Hawk bird with square body, wings and tail rectangles	BSS 8.1-8.18 9.2-9.10
Vakrapksavyasta-puccahasyenacit	Hawk bird with bent wings and out spread tail	BSS 10.1-10.14 11.2-11.13

Kankacit	Hawk bird with curved wing and tail	BSS 12.1-12.8
Alajacit	Alaja bird with curved wings and tail	BSS 13.1-13.5
Praugacit	Isosceles triangle	BSS 14.4-14.8
Ubhayata praugacit	Rhombus	BSS 15.2-15.6
Rathacakracit	Chariotwheel	BSS 16.3-16.5, 16.6-16.20
Dronacit	Square trough	BSS 17.1-117.12
	Circular trough	BSS 18.1-18.15
Samasanacit	Isosceles trapezium	BSS 19.1-19.11
Kurmacit	Tortoise	BSS 20.1-20.21 21.2-21.13
Vedi		
Māhavedi/ Saumikyāvedi	Isocels trapezium (face a =, base $b = 30$, height c = 36)	BSS 4.3 [2] & ([3], p.69) ([7], p.22)
Sautramanivedi	Isosceles trapezium ($a = 8\sqrt{3}$, $b = 10\sqrt{3}$, $c = 12\sqrt{3}$)	BSS 3.12, 1.85
Paitrkivedi	Isoceles trapezium ($a = 8$, $b = 10$, $c = 12$) Square having four corners in four corrdinate directions.	BSS 1.82, 3.11 ([17], p. 68)
Ultaravedi	Four cornered and each side is 36 angulas	BSS 3.10
Asvamedhavedi	Isosceles trapezium (a = 32, b = 40, c = 54)	

(Units are in padas otherwise stated)

Following are some of the geometrical methods which are used for construction of altars.

1. Construction of a trapeziam Similar to a given trapezium.
2. Construction of trapezium similar to double of its area.
3. Construction of a trapezium similar to third part of its area.
4. Construction of an isosceles trapezium similar to given trapezium by increasing/diminishing its sides by a certain proportion.
5. Construction of an isosceles trapezium of given area.
6. Construction of a square of given area
7. A quadrilateral formed by the lines joining the middle parts of the sides of a square is a square whose area is half that of the original one.
8. A quadrilateral formed by the lines joining the middle points of the sides of the rectangle is a rhombus whose area is half that of the rectangle.

   In addition ot geometical problems. Sulb Sutras also concern extensively with the use of instruments such as bomboo rod, geometrical compass, peg, rope and sanku for altar construction; relative distance between Āhavaniya, Daksināgni, Gārhapatya and Uttaravedi; relative position of various altars with regard to Māhavedi; enlargement of fire altars; a few indeterminate problems; a little of geometrical algebra; fractions; elementry concept of arithmetic series [13].

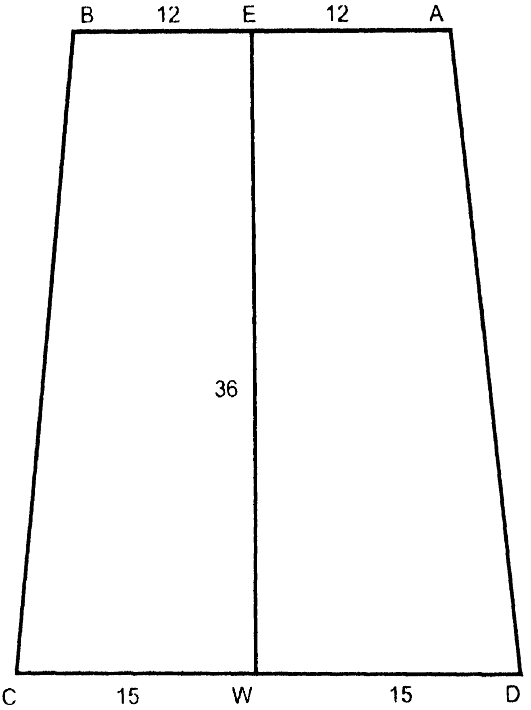
MAHĀVEDI

त्रिꣳशत्पदानि प्रक्रमा वा पश्चात्तिरश्ची भवति षट्त्रिꣳशत्प्राची
चतुर्विꣳशतिः पुरस्तात्तिरश्चीति महावेदेर्विज्ञयाते मानयोगस्तस्या व्याख्यातः॥ (BSS 1.90)

The meaning of above Sūtra is as follows:

"The western side is thirty padas of prakrams long, the prācī thirty-six the eastern side twenty four" this is the tradition for Mahāvedi (the vedi used at the soma sacrifices); it has been explained how it is to be measuued out [17].

In other words above may be stated as Mahāvedi described as isosceles trapezium measuring 80 padas or prakarmas on western side, 36 padas along east-west line 24 padas on eastern side [13].



4 padas = 1 cm

(see 13)

Description :

Mahāvedi ABCD is isosceles trapezium of width 36 and parallel sides 24 and 30 units. This is constructed by first making the central-axis (east-west line), points W and E and then drawing the four segments WD, WC, EA and EB along the other cardinal directions (south and north) ([13], p.22) Mahavedi is called Saumiki vedi in ASS.

Calculation of area of Mahavedi by Śulba Sūtra method:

Draw a perpendicular form A on side CD and from C on AB. Let perpendiculars meet on WD at G and on AB at H.

Area of Mahavedi ABCD = Area of rectangle AGCH.

E and W are middle points of sides AB and CD.

$\therefore$ BE = EA = 12

$\therefore$ CW = WD = 15

$\therefore$ WG = 12 and CG = 15+12 = 27

$\because$ EW = 36

$\therefore$ AG = HC = EW = 36

$\therefore$ Area of rectangle AGCH = CGxCH = 27x36 = 972 sq. padas [4].

Hence area of Mahavedi = 972 sq. padas.

Verificaton of area by modern method:

$$\text{Area of Mahavedi} = \frac{1}{2}\text{ (Sum of parallel sides) x height}$$

$$= \frac{1}{2}(30+24)\times 36$$

$$= 54\text{x}18 = 972 \text{ sq. padas.}$$

Length of non-parallel sides:

Since EW = AG

$\therefore$ AG = 36

and EA = WG = 12

$\therefore$ GD= WD-WG=15-12=3

Now in triangle AGD, using Sulbkara's theorem

$AD^2=AG^2+GD^2$

$\therefore$ $AD^2 = (36)^2 + (3)^2$

$= 1296+9=1305$

$\therefore$ $AD=\sqrt{1305}$ =36.12

$\therefore$ AD=BC=36.12 padas.

SAUTRĀMANIVEDI

वेदितृतीये यजेतेति सौत्रामणिकी वेदिमभ्युपदिशन्ति॥ (BSS 1.85)

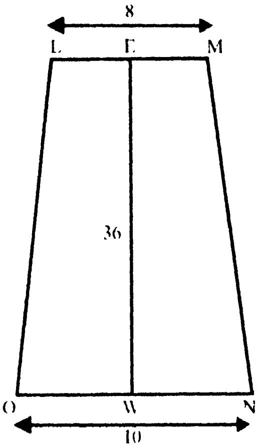
"He is to sacrifice on the third part of the vedi" thus it explains the formation of vedi of the Sautramani sacrifice.

महावेदेस्तृतीयेन समचतुरस्रकृताया अष्टादशपदा पार्श्वमानी भवति॥ (BSS 1.86)

The side of that area which is formed into a square with the third part of the Mahavedi is eighteen padas long.

तस्यै दीर्घकरण्यामेकतोऽणिमत्करण्यां च याथाकामीति ॥ (BSS 1.87)

The meaning of the sutra is : instead of giving to the Sautrāmanivedi the form of a square we may give to it the form of an "oblong which is shorter on one side" and the area of which is equal to the third part of the Mahavedi [17]..



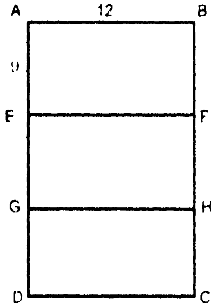
Description :

The Sautrāmanivedi is constructed exactly in the same manner as the Mahavedi. Measures prescribed for the Mahavedi are 36 padas for prāci (east-west line), 30 padas for western side, 24 padas for eastern side all these lengths were devided by three then we obtain 12 padas for prācī, 10 padas for western side and 8 padas for estern side. The area of figure formed is one ninth of the area of Mahavedi which is 108 units of area; Multiplying it by 3, we get 324 units of area which is equal to the area of Sautramanivedi and dimensions of Sautramani vedi are as follows 8,10 and 36 padas [4].

Measurment of area of Sautramanivedi by Sulba Sutra method:

AB = 12 Padas

AE = EG = GD = 9 padas



∴ AF = FG = HD = 108 sq. padas

∴ AC = 324 sq. padas.

Area of rectangle ABFE = 12X9 = 108 sq. padas

∴ Area of rectangle ABCD = 12X27 = 324 sq. padas

Area of Sautramanivedi LMNO = Area of rectangle ABCD = 324 sq. padas [4].

Varification of area by modern method :

Area of Sautramanivedi LMNO $= \frac{1}{2}$ (Sum of parallel sides) x height

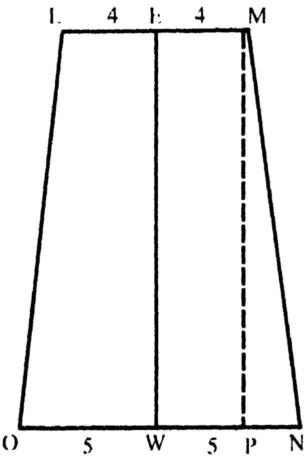
$= \frac{1}{2} (8+10) \times 36$

$= \frac{1}{2} \times 18 \times 36 = 9 \times 36 = 324$ sq. padas

Length of non-parallel sides:

From fig., MP is perpendicular on N.

Since E and W are middle points of sides LM and ON



∴ LE = EM = 4 and OW=WN=5

∴ PN = WN-WP

= 5-4 = 1 (∵ WP = EM)

Since EW = 36

∴ MP = 36

In triangle MPN, using Sulbkara's theorem

$MN^2 = MP^2 \quad PN^2$

$= (36)^2 + (1)^2$

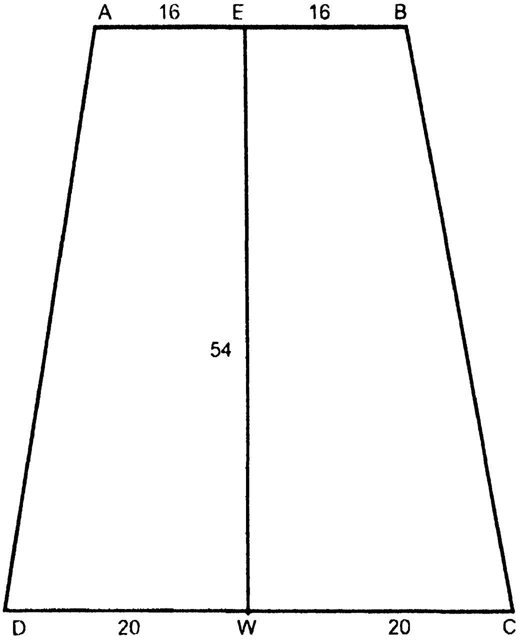
MN = $\sqrt{1297}$ =36.01 padas

MN = LO=36.01 padas

AŚVAMEDHA VEDI

द्विस्तावा वेदिर्भवतीत्यश्वमेधे विज्ञायते॥ (ASS 5.36)

Area of the altar for Aśvamedha is double the area of Saumikivedi (Mahavedi) i.e. 1944 sp. pada [4].



Description :

Because area of Mahavedi is 972 sq. padas therefore area of Aśvamedhavedi will be 1944 sq. padas.. Shape of this vedi is the same as that of Mahavedi. Accordingly prācī of Aśvamedha vedi is 54, western side is 40 and eastern side is 32 padas [4].

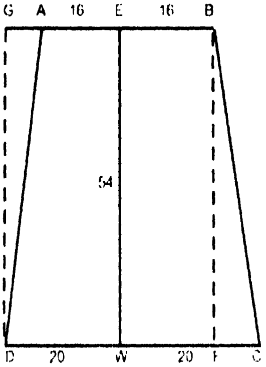
Now in figure ABCD is Aśvamedhavedi.

Measurement of area of Asvamedhavedi by Sulba Sutra Method :

Draw two perpendicular one from B on DC and second from D on BA. Let perpendiculars meet on WC at F and on BA at G.

Area of rectangle BFDG= Area of Aśvamedhavedi ABCD

Since E and W are middle points of the sides AB and DC.



∴ AE=EB=16 and DW=WC=20

also GD=EW=BF=54

∴ DF=DW + WF = 20 + 16 = 36

∵ Area of rectangle BFDG = 54 x 36 = 1944 sq. padas

Hence area of Aśvamedhavedi = 1944 sq. padas.

Varification of area by modern method :

Area of Asvamedhavdi $=\frac{1}{2}$ (Sum of parallel sides) x height

$$=\frac{1}{2}(32+40)\times 54$$

$$=\frac{1}{2}\times 72\times 54=36\times 54=1944 \text{ sq. padas}$$

Length of non-parallel sides:

from fig., FC = WC - WF = 20-16 = 4

and BF = 54

In triangle BFC, using Sulbkara's theorm

$BC^2 = NF^2 + FC^2$

$BC^2 = (54)^2 + (4)^2$

$= 2916 + 16 = 2932$

∴ $BC = \sqrt{2932} = 54.15$

∴ BC = AD = 54.15 padas.

In this paper the dimensions of Mahavedi, Sautramanivedi and Aśvamedhavedi and their shapes are described. Their areas have been calculated by Śulba Sūtra method and modern method. In each case area comes out to be equal, showing the exactness of the geometrical methods described in Śulba Sūtra.

REFERENCES

1. Archaeological Survey of India, Annual Report, (ASI, AR), 1920-21, P.17
2. Afzal Ahmad : On the π of Aryabhata I, Ganita Bharati (GB), 3 (1981) 83-85.
3. सत्यप्रकाश एवं पं0 रामस्वरूप शर्मा (सम्पादक), आपस्तम्ब शुल्बसूत्रम्, प्राचीन वैज्ञानिकाध्ययन अनुसंधान संस्थान, नई दिल्ली, (1938)

4 दामोदर झा (संपादक), आपस्तम्ब-शुल्बसुत्रम्, विश्वेश्वरानन्द वैदिक शोध संस्थान, होशियारपुरम् (पंजाब) (1988).

5. Bodhāyanācār and Bhagıratha Prasada Trıpathi 'Vagisasatri' (Ch. Ed.): Baudhāyana Śulba Sutrām, Research Institute, Sampurnanad Sanskrit Visvavidyalaya, Varanasi, 1979.
6. A.K. Bag, : Mathematics in Ancient and Medieval India, Chaukhamba Orientailia, Varanasi, 1979.
7. B. Datta, : The Science of Śulba, A study in Early Hindu Geometery; Calcutta University, 1932.
8. B. Datta and A.N. Singh : History of Hindu Mathematics (Part I and II), Asia Publishing House, Bombay, 1962.
9. R.C. Gupta, : Vedic Mathematics from the Śulba Sūtras; J. of Mathematics Education 9 (2) (1989) 1-10.
10. Nidhi Handa, Kātyāyana Śulba Sūtras and Modern Mathematical Interpretations of its Sutras, Ph.D. Thesis, Gurukul Kangri Uni, Haridwar.
11. Pt. Lashan Lal Jha. (Ed.) : Lilāvati of Bhāskarācārya II, The Chowkhamba Vidya Bhawan, Varanasi Chowk 1961.
12. S.D. Khadikar (Ed.): Kātyāyana Śulba Sūtras, Vaidika Samsodhana Mandoler, Poona, 1974.

13. Vinod Mishra: A Study of Vedic Geometry and its Relevance to Science and Technology, Ph.D. Thesis, Gurukul Kangri Uni, Haridwar (1977).

14. S.N.Sen: A Bibliography of Sanskrit works on Astronomy and Mathematics part I, National Institute of Sciences of India, New Delhi, 1966.

15. V.K. Sharma and Yogita Bana : Some Geometrical Construction form Baudhayana Sulba Sutra, J. Natural & Phys. Sci. 18 (1) (2004) 57-67.

16. Sharma, V.K. and Yogita Bana : Karani and related Construction from Baudhāyana Śulba Sūtras, ACCST Research Journal, 11 (2004) 4-10

17. D.N.Yajvan and G. Thibout : Baudhayana Sulba Sutram, The Research Institute Ancient Scientific Studies, New Dehli, 1968.

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 47-54

KINETICS AND MECHANISM OF MN (II) CATALYSED PERIODATE OXIDATION OF O-CHLORO ANILINE IN AETONE-WATER MEDIUM

R.D.KAUSHIK*, YOGESH C. NAINWAL, VIRENDRA SINGH AND A.K. CHAUBEY

(Received 10.08.2004)

ABSTRACT

Kinetic-mechanistic studies made for periodate oxidation of o-chloroaniline (OCA) in acetone-water medium, have been used for derivation of rate law and proposing mechanism which satisfy various observations like first order in each reactant and catalyst, stoichiometry (1 mol OCA : 2 mol periodate), kinetic parameters (Ea = 6.5 K cals/mol; A = 1.45 $\times 10^5$ lit/mol/sec; $\Delta S^{\neq}$ = −38.0 EU; $\Delta F^{\neq}$ =17.72 K Cals/mol; $\Delta H^{\neq}$ = 5.9 K Cals/mol), main product identified (2-chloro-1,4 benzoquinone), and the effect of pH, dielectric constant, and ionic strength on reaction rate.

Key words : Kinetics and mechanism, Mn (II) catalyst, Periodate oxidation, o-chloroaniline, 2-chloro-1,4-benzoquinone.

INTRODUCTION

Reports on kinetic-mechanistic studies on the Mn (II) catalysed periodate oxidation of aromatic amines are very few[1-4]. In continuation to our earlier communications on uncatalysed oxidations of such type [5-7], present paper deals with the kinetic studies made on Mn (II) catalysed periodate oxidation of OCA in acetone-water medium.

MATERIALS AND METHODS

Chemicals of E.Merck/CDH A.R. grade were used after distillation/recrystallization. Triply distilled water was used for preparation of the solutions. The progress of the reaction was followed spectrophotometrically [5] by recording the absorbance at 475 nm, i.e. the λ_{max} of reaction mixture in the duration in which the λ_{max} did not change. The pH was maintained at 5.0 by using Thiel. Schultz and Koch buffer [6,7]

Department of Chemistry, Gurukul Kangri University, Haridwar-249404 (Uttaranchal) India

in all kinetic runs except when the effect of pH was studied, Plane mirror method and Guggenheim's method were used for evaluation of initial rates $[(dA/dt)_i]$ and pseudo first order rate constant k_1 or second order rate constant k_2 respectively. NaCl solutions were used for maintaining the ionic strength (μ) in the kinetic runs.

RESULTS AND DISCUSSION

1 mol OCA consumed 2 moles of periodate as determined iodometrically. The data (table-1) indicated 2nd order for the reaction, being first order in each reactant. Linear relation between concentration of the reactants and rate supported the 2nd order kinetics. In pseudo first order conditions (table-2), the $[(dA/dt)]^{-1}$ or k_1^{-1} vs$[S]^{-1}$ Plots were linear with almost negligible intercept, suggesting the Michaelis-Menten type kinetics being followed with respect to both reactants with the possiblity of formation of a fast decaying intermediate complex between reactants [10,11]. Data in table-3 established the 1st order in catalyst. Rate-pH profile showed a maxima at pH=5.0 (table-4). A linear relation between log (dA/dt)i or log k_2 and 1/D with negative slope (where D is the dielectric constant of the medium) and a primary linear type plot between log $(dA/dt)_i$ or log k_2 vs ionic strength (μ) that were obtained from the data in table-5, indicated an ion-dipole interaction in this reaction. Arrhenius plot was made between 30 ± 0.1°C to 45 ± 0.1°C and the values of different thermodynamic parameters evaluated taking $[OCA]=1.0 \times 10^{-3}M$, $[NaIO_4]=1.0 \times 10^{-4}M$, $[Mn^{++}]=8.0 \times 10^{-6}$ M and acetone = 10.0% (v/v), are Ea=6.5 K cals/mol; A=1.45 x 10^5 lit/mol/sec ; $\Delta S^* = -38.0$ EU ; $\Delta F^* = 17.72$ K cals/mol ; $\Delta H^* = 5.9$ K cals/mol. Low value of energy of activation and high frequency factor are characteristic of a bimolecular reaction in the solution in the which the reacting species are larger in size. A large negative value of ΔS^* suggests the formation of strongly solvated, charged and rigid transition state.

Table-1. Determination of order w.r.t. reactants.

λ_{max} = 475 nm; pH= 5.0; Acetone = 10.0% (v/v); Temp = 30 ± 0.1°C;

[Mn^{++}]=8.0 x 10^{-6}M (6x10^{-6} M for variation of [$NaIO_4$])

[OCA] x 10^4M	2.0	4.0	6.0	8.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0
[$NaIO_4$]x10^3M	1.0	1.0	1.0	1.0	1.0	0.1	0.2	0.3	0.4	0.8	1.0
(dA/dt)x10^2(min^{-1})	3.0	5.6	8.0	10.3	12.4	7.0	14.0	27.0	40.0	54.0	64.0

Table-2. λ_{max} =475 nm; pH=5.0; Acetone = 10.0%(v/v); Temp = 30.0 ± 0.1°C;

[Mn^{++}]=8.0x10^{-6} M (6x10^{-6} M for variation of [$NaIO_4$]).

[OCA]x10^3M	1.0	1.2	1.4	1.6	1.8	0.1	0.1	0.1	0.1	0.1	0.1
[$NaIO_4$]x10^3M	0.10	0.10	0.10	0.10	0.10	1.0	2.0	3.0	4.0	5.0	6.0
$(dA/dt)_i$x10^3(min^{-1})	6.5	14.0	21.0	27.5	32.5	7.0	15.0	22.0	26.0	34.0	40.0
k_1x10^3(sec-1)	1.73	2.01	2.30	2.50	2.78	1.54	2.30	3.07	3.45	3.84	4.22

Table-3. Determination of order w.r.t. Mn^{++}

[$NaIO_4$]=1.0x10^{-4}M;[OCA]=1.0x10^{-3}M;Temp. = 30.0 ± 0.1°C;

Acetone =5.0% (v/v); λ_{max} =475 nm

[Mn^{++}] x10^{-6}M	4.0	6.0	8.0	10.0	12.0	14.0
$(dA/dt)_i$x10^3(min^{-1})	6.5	10.0	13.0	16.0	19.0	22.0

Table-4. Rate-pH profile.

[OCA]=1.0×10^{-3} M; [$NaIO_4$] = 1.0 x 10^{-4} M; λ_{max} =475 nm; Acetone = 5.0% (v/v);

Temp. = 30.0 ± 0.1°C; [Mn^{++}] = 8.0x10^{-6}M

pH	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
$(dA/dt)_i$x10^3(min^{-1})	5.0	5.25	6.5	8.0	5.5	5.0	4.5	4.0	3.0

Table-5 Effect of D and μ on reaction rate.

[OCA]= 1.0×10^{-3} M; [$NaIO_4$] =1.0×10^{-4} M; λ max= 475 nm; * Acetone= 5.0% (v/v); Temp.= 30.0 ± 0.1°C; [Mn^{++}]= 8.0×10^{-6} M.

D	72.4	71.0	69.7	68.4	--	--	--	--
* $\mu \times 10^2$	--	--	--	--	10.0	12.3	14.1	17.3
$(dA/dt)_i\ 10^3\ (min^{-1})$	9.0	8.0	7.0	6.0	6.5	7.0	7.5	8.25
$k_1 \times 10^3 (Sec^{-1})$	3.65	2.40	1.73	1.34	1.54	1.60	1.63	1.73
k_2 (1. mol^{-1}.(Sec^{-1})	3.65	2.40	1.73	1.34	1.54	1.60	1.63	1.73

The increase in the rate from pH 3.5 to 5.0 may be due to the decrease in the protonation of OCA from pH 3.5 to 5.0 (table-4) which makes greater concentration of OCA available for the reaction. This leads to the assumption that unprotonated OCA is the reactive species in present case. Second part of this profile suggests that the periodate monoanion [IO_4^-] is the reactive species of periodate [9] whose concentration goes on decreasing with increase in pH beyond the value 5.0 [9] decreasing thereby the rate of reaction beyond pH 5.0.

Based on these results, the mechanism may be proposed (Chart) involving the lone pair of electrons on nitrogen atom of OCA for the co-ordinate bond formation between OCA and Mn^{++} species in a reversible step to from complex C_1 [step (i)] which, in turn, interacts with IO_4^- in slow and rate determining step (ii) to give C_2. C_2 Changes by fast hydrolysis in to C_3. The formation of a charged intermediate complex C_2 by the attack of IO_4^- on the nitrogen of anilino group and stabilization of positive charge on nitrogen of this group, have also been observed and supported by LFER studies for the uncatalysed periodate oxidation of few aromatic amines [8] In addition, a high negative value of entropy of activation and the effect of dielectric constant on the reaction rate support the involvement of solvation effects in this reaction.

It should also be noted that the initial part of the reaction is significant in the present case and the second molecule of IO_4^- reacting later to give C_4 is not significant. C_4 changes by fast hydrolysis to give C_5 i.e. the main product of raction that has been isolated and seperated by employing the method reported eariler in case of other aromatic amines [6,7]. It was characterized as 2-chloro-1,4- benzoquinone on the basis of test for quinone [4], M.P. [12], UV-VIS, IR and H-nmr spectra (showing characteristic bands for C-Cl [13] and benzoquinones [14]). The overall process may be representead as follows:

$$\mathrm{OCA} + \mathrm{Mn}^{++} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C_1 \text{--------------------(i)}$$

$$C_1 + [IO_4] \xrightarrow[slow]{k_3} C_2 + Mn^{++} \text{--------------------(ii)}$$

$$C_2 \xrightarrow[H^+, H_2O]{Fast} C_3 + H_2O + HIO_3 \text{--------------------(iii)}$$

$$C_3 + IO_4^- \xrightarrow{+H^+} C_4 + H_2O + HIO_3 \text{----------------------------------(iv)}$$

$$C_4 + \xrightarrow[Fast]{+H_2O} C_5(2-\text{chloro}-1{,}4-\text{benzoquinone}) \text{-------------(v)}$$

On applying steady state treatment to C_1, the rate law in terms of rate of loss of $[IO_4]$ may be derived as follows:

Rate of loss of $[IO_4]$ or-d $[IO_4^-]/dt = k_3 [C_1] [IO_4^-]$

= Rate of loss of C_1 or- d $[C_1]/dt$ -------------(i)

Rate of formation of C_1=+d$[C_1]/dt$=$k_1[OCA][Mn^{++}] - k_2 [C_1]$

At steady state,- d $[C_1]/dt$ = + d$[C_1]/dt$

Therefore, $k_3 [C_1] [IO_4^-] = k_1 [OCA] [Mn^{++}] = k_2 [C_1]$

or $[C_1] \{k_3. [IO_4^-] + k_2\} = k_1 [OCA] [Mn^{++}]$

$$\text{or } [C_1] = \frac{k_1[OCA][Mn^{++}]}{k_3[IO_4^-]+k_2} \text{---------------------(2)}$$

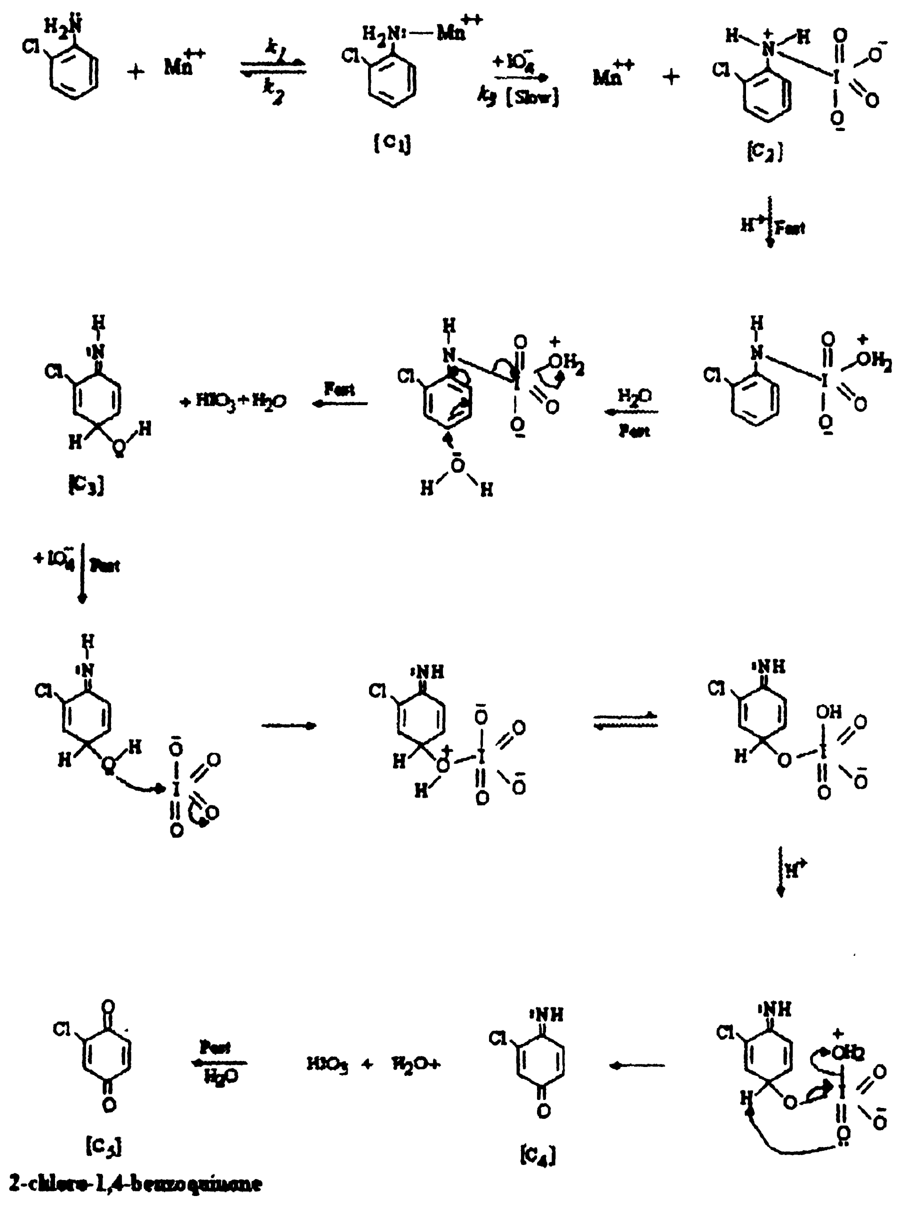
$$\text{From (1) and (2),- d } [IO_4^-]/dt = \frac{k_3.k_1.[OCA][Mn^{++}][IO_4^-]}{k_2 + k_3[IO_4^-]} \text{----------(3)}$$

Since step (ii) is slow and rate determining step, hence $k_3 [IO_4^-] << k_2$ may be assumed. Therofore, the rate law (3), may be written as rate law (4) which explains all the kinetic results.

$-d[IO_4^-]/dt = k_{obs}.[OCA][Mn^{++}] [IO_4^-]$------------------(4)

where, $k_{obs} = k_3 k_1 / k_2$

CHART


k_1
k_2
$+IO_4^-$
k_3 [Slow]
[C1]
[C2]
H+ Fast
$+ HIO_3 + H_2O$
Fast
H_2O
Fast
[C3]
$+IO_4^-$ Fast
H+
Fast
H_2O
$HIO_3 + H_2O +$
[C5]
[C4]
2-chloro-1,4-benzoquinone

REFERENCES

1. I.F. Dolmanova, V.P. Poddubienko, and V.M. Peshkova: Determination of manganese (II) by kinetic methods using the oxidation of p-phenetidine, Zh. Anal. Khim. 25 (11) (1970), 2146-2150.

2. R.D. Kaushik, A.K. Chaubey and R.P. Singh: A new kinetic-spectrophotometric method for the determination of Mn (II) in water, Indian J. Environ. & Ecoplann. 7 (1) (2003), 29-34.

3. R.D. Kaushik, Amrita and Sumitra Devi : An improved method for nanogram determination of Mn (II) in aqueous medium, J. Curr. Sci. 3 (1) (2003), 197-202.

4. R.D. Kaushik, Shashi, Sumitra Devi and R.P. Singh: Simple kinetic-spectrophotometric method for the determination of Mn (II) in water, Asian J.Chem 16 (2) (2004), 837-840.

5. R.D. Kaushik et al.: Asian J.Chem., 11 (2) 633 (1999); 12 (4), 1123, 1129, 1229 (2000).

6. R.D. Kaushik, R.P. Singh and Shashi: Kinetic-mechanistic study of periodate oxidation of p-chloroaniline, Asian J. Chem 15 (3 & 4) (2003), 1655-1658.

7. R.D. Kaushik, A.K. Chaubey and P.K. Garg: Kinetics and mechanism of periodate oxidation of p-phenetidine, Asian J.Chem. 15 (3 & 4((2003), 1485-1490.

8. S.P. Srivastava, V.K. Gupta M.C. Jain, M.N. Ansari and R.D. Kaushik: Ther modynamic and LFER studies for the oxidation of anilines by periodate ion, Thermo Chimica Acta, 68 (1983), 27-33.

9. V.K. Pavolva, Ya. S. Sevchenko and K.B. Yatsmiriskii: Kinetics and mecha nism of oxidation of diethylaniline by periodate, Zh.Fiz. Khim., 44 (3) (1970), 658-63.

10. R.D. Kaushik, Shashi, Amrita and Sumitra Devi: Kinetics and mechanism of Mn (II) catalysed periodate oxidation of 4-chloro-2-methylaniline, Asian J. Chem. 16 (2) (2004), 818-822.

11. N. Nalwaya, A. Jain and B.L. Hiran : Kinetics of oxidation of glycine by pyridinium bromo chromate in acetic acid medium, J. Indian Chem. Soc. 79 (7) (2002), 587-589.

12. J. Buckingham (Ed.): Dictionary of Organic compounds, 5th edn., Vol. I, Chapman & Hall, New York, 1982.

13. J.R. Dyer : Application of absorption spectroscopy to Organic compounds, Prentice-Hall of India, New Delhi, 1984 p 38.

14. R.M. Silverstein : Spectrometric identification of organic compounds, 5th ed., John Wiley and sons. Inc. New York 1991.

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 55-76

STRONG SUMMABILITY OF FUNCTIONS BASED ON (D,α,β) (C,ℓ,m) SUMMABILITY METHODS

B.P. MISHRA* & S.K. SINGH*

(Received 10.08 2004)

ABSTRACT

Singh [7] defined strong summability methods. Here, we have defined summability of function based on $(D,\alpha,\beta),(C,\ell,m)$ summability methods and investigated some of their properties.

Key words and Pharase: Strong summability, function, $(D,\alpha,\beta),(C,\ell,m)$ summability methods, Lebesgue integral, Holder's inequality and Minkowski's inequality.

Mathematics subject classifications (2000) : 40F05.

SOME NOTATIONS AND DEFINITIONS

Let $s(x)$ be any function which is Lebesgue integrable in $(0,x)$ for all finite $x > 0$ and that $s(x)$ is bounded in some right hand neighbourhood of the orgin. Integrals of the form $\int_0^\infty$ are throughout to be taken as $\lim\limits_{X\to\infty}\int_0^X$, $\int_0^X$ being taken as a Lebesgue integral. We write

$$\partial_{\ell,m}(X)=\begin{cases} S(x) & l=0 \\ \left\{\dfrac{\Gamma(\ell+m+1)}{\Gamma(\ell)\Gamma(m+1)}\right\}\dfrac{1}{X^{\ell+m}}\displaystyle\int_0^x (x-y)^{\ell-1}y^m s(y)dy, & \ell>0, m>-1 \end{cases} \tag{1.1}$$

We also write

$$g^{\alpha,\beta}(y)=\left\{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\right\}y^{\beta+1}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}s(x)dx, \tag{1.2}$$

if this exists. We also write

$$U_{\alpha,\beta,\ell,m}(y)=\left\{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\right\}y^{\beta+1}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx, \tag{1.3}$$

if this exists, where $\ell>0, m>-1, y>0, \beta>-1, \alpha>0$.

---

* Department of Mathematics & Statistic D.D.U. Gorakhpur University, Gorakhpur-273009

With the usual terminology, we say that *s(x)* is summable

(i) (C,ℓ,m) to the sum s, if $\partial_{\ell,m(x)}(x)$ exists, and tends to a limit s as $x \to \infty$,

(ii) $[C,\ell,m]_p$ or strongly summable (C,ℓ,m) with index p to the sum s, where

$\ell \geq 1, m \geq -1, p \geq 1$, if

$$\int_0^x |\partial_{\ell-1,m}(y)-s|^p dy = o(x) \text{ as } x \to \infty \tag{1.4}$$

(iii) (D,α,β) to the sum s, it $g^{\alpha,\beta}(y)$ tends to s as $y \to \infty$,

(iv) (D,α,β) (C,ℓ,m) to the sum s, if $U_{\alpha,\beta,\ell,m}$ (y) tends to s as $y \to \infty$.

Definition (ii) & (iii) are due respectively to Mishra and Srivastava [5] and Kwee [3], while (iv) is due to Mishra and Singh [6].

Let

$$s(x) = \int_0^x a(y)dy, \tag{1.5}$$

where a (y) is Lebesgue integrable in (0, x) for any finite $x > 0$. Since $U_{\alpha,\beta,\ell-1,m}(y)$ converges for all $y > 0$, then it is easy to justify the differentiation under integral sign ($\int$) and that

$$y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y) = -(\ell+m)[U_{\alpha,\beta,\ell,m}(y) - U_{\alpha,\beta,\ell-1,m}(y)] \tag{1.6}$$

$$= B_{\alpha,\beta,\ell,m(y)} \qquad (\ell \geq 1),$$

where

$$B_{\alpha,\beta,\ell,m}(y) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}} \tau_{\ell,m}(y)dy.$$

Also we know that

$$y\frac{d}{dy}\partial_{\ell,m}(y) = (\ell+m)[\partial_{\ell-1,m}(y) - \partial_{\ell,m}(y)] = \tau_{\ell,m}(y) \qquad (1.7)$$

$$= \frac{\Gamma(\ell+m+1)}{\Gamma(\ell)\Gamma(m+1)}\frac{1}{y^{\ell+m}}\int_0^y (y-v)^{\alpha-1}v^m v a(v)dv.$$

Before giving the justification of (1.6) we first make some remarks on (1.7). If we do not assume (1.5), then $\tau_{\ell,m}(y)$ has not been defined, so that the second equation (1.7) is meaningless. However the first equality holds every where if $\ell \geq 2$ and almost every where if $2 > \ell \geq,1$. if we assume (1.5), then both the equations in (1.7) hold every where for $\ell \geq 1$. $y \geq 1$, the middle term in (1.7) is not defined. However, the equality

$$y\frac{d}{dy}\partial_{\ell,m}(y) = \tau_{\ell,m}(y)$$

continues to hold almost every where for $1 > \ell > 0$.

In order to get a valid proof of (1.6), under the assumption that the integral defining $U_{\alpha,\beta,\ell-1,m}(y)$ converges for all $y > 0$, that it is necessary and sufficient for the convergence of (1.3), that

$$\int_1^\infty \frac{\partial_{\ell-1,m}(x)}{x^{\beta+2}}dx \text{ should converge} \qquad (1.8)$$

We write

$$R_{\ell-1,m}(y) = \int_y^\infty \frac{\partial_{\ell-1,m}(x)}{x^{\beta+1}}dx$$

which is bounded for $y \geq 0$ and tends to zero as $y \to \infty$.

Now

$$\partial_{\ell,m}(u) = \frac{\ell+m}{u^{\ell+m}}\int_0^u v^{\ell+m-1}\,\partial_{\ell-1,m}(v)dv$$

$$= -\frac{(\ell+m)}{u^{\ell+m}}\int_0^u v^{\ell+m-1}v^{\beta+R}dR_{\ell-1,m}(v)$$

$$= -\frac{(\ell+m)}{u^{\ell+m}}\int_0^u v^{\ell+m+\beta+1}dR_{\ell-1,m}(v) \tag{1.9}$$

Since $R_{\ell-1,m}(v) \to \infty$, therefore given $\in >0$, there is number u_0 such that $|R_{\ell-1,m}(v)|$ $< \varepsilon$ for $v > u_0$, we also note that $|R_{\ell-1,m}(v)| = O(1)$ for $v < u_0$. Hence we see from (1.9), on integration by parts, that

$$\partial_{\ell,m}(u) = -\frac{\ell+m}{u^{\ell+m}}\left\{v^{\ell+m+\beta+1}R_{\ell-1,m}(v)\right\} + \frac{\ell+m}{u^{\ell+m}}\int_o^u \frac{d}{dv}(v^{\ell+m+\beta+1})R_{\ell-1,m}(v)dv$$

$$= -\frac{\ell+m}{u^{\ell+m}}\int_o^{u'} \frac{d}{dv}(v^{\ell+m+\beta+1})R_{\ell-1,m}(v)dv + \frac{\ell+m}{u^{\ell+m}}\int_{u'}^u \frac{d}{dv}(v^{\ell+m+\beta+1})R_{\ell-1,m}(v)dv$$

$$= O\left(u^{\beta+1}\right) + o\left(u^{\beta+1}\right)$$

$$= o\left(u^{\beta+1}\right), \text{ if } u \to \infty$$

Hence $\partial_{\ell,m}(u) = O\left(u^{\beta+1}\right)$, as $u \to \infty$ (1.10)

Let $H_{\alpha,\beta,\ell,m}(y) = \int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$

$$H_{\alpha,\beta,\ell,m}(\eta+y) = \int_0^\infty \frac{x^{\alpha-1}}{(x+\eta+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$$

Let $y > y_0$ to be fixed . By Lagrange's for mula, for the remainder in Taylor's series, we have, for $\eta > -y$, and for all $x \geq 0$

$$\frac{1}{\eta}\left\{H_{\alpha,\beta,\ell,m}(\eta+y) - H_{\alpha,\beta,\ell,m}(y)\right\} = \frac{1}{\eta}\int_0^\infty x^{\alpha-1}\partial_{\ell,m}(x)\left\{\frac{1}{(x+n+y)^{\alpha+\beta+1}} - \frac{1}{(x+y)^{\alpha+\beta+1}}\right\}dx$$

By Lagrange's formula, we have

$$\frac{1}{\eta}\left\{\frac{1}{(x+\eta+y)^{\alpha+\beta+1}}-\frac{1}{(x+y)^{\alpha+\beta+1}}\right\}=-\frac{(\alpha+\beta+1)}{(x+y)^{\alpha+\beta+2}}+\frac{\eta}{2}\frac{(\alpha+\beta+1)(\alpha+\beta+2)}{(\xi+y)^{\alpha+\beta+3}}$$

for some ξ between $(x+y)$ and $(x+\eta+y)$ (hence ξ depends on x,y,η). Now choose a fixed y' with $0<y'<y$. Then, for $\eta>-(y-y')$ (and thus, in particular, for all sufficiently small η), we have $\xi\geq y$,

$$\left|\frac{1}{\eta}\{H_{\alpha,\beta,\ell,m}(\eta+y)-H_{\alpha,\beta,\ell,m}(y)\}+\int_0^\infty\frac{(\alpha+\beta+1)x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}}\partial_{\ell,m}(x)dx\ \right|$$

$$\leq\frac{|\eta|}{2}\ \left|\int_0^\infty x^{\alpha-1}\partial_{\ell,m}(x)\frac{(\alpha+\beta+1)(\alpha+\beta+2)}{(x+y)^{\alpha+\beta+3}}dx\ \right|$$

$$\leq o\,(|\eta|).$$

Since the integral on the right hand side converges. Hence $H_{\alpha,\beta,\ell,m}(y)$ is differentiable, and

$$\frac{d}{dy}H_{\alpha,\beta,\ell,m}(y)=\lim_{\eta\to 0}\frac{1}{\eta}[H_{\alpha,\beta,\ell,m}(\eta+y)-H_{\alpha,\beta,\ell,m}(y)]$$

$$=-(\alpha+\beta+1)\int_0^\infty\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}}\partial_{\ell,m}(x)dx. \qquad (1.12)$$

We know that

$$U_{\alpha,\beta,r,m}(y)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\,y^{\beta+1}\int_0^\infty\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$$

Differentiating this equation, we have

$$\frac{d}{dy}U_{\alpha,\beta,r,m}(y)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\Big[(\beta+1)y^{\beta}\int_0^\infty\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$$

$$-\,y^{\beta+1}\int_0^\infty\frac{x^{\alpha-1}(\alpha+\beta+1)}{(x+y)^{\alpha+\beta+2}}\partial_{r,m}(x)dx\Big]$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}[y^{\beta}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\{(\beta+1)-\frac{y(\alpha+\beta+1)}{(x+y)}\}\partial_{\ell,m}(x)dx]$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}[y^{\beta}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\{(\beta+1)-\frac{y(\alpha+\beta+1)}{(x+y)}\}\partial_{\ell,m}(x)dx]$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}}\{x(\beta+1)-y\alpha\}\partial_{\ell,m}(x)dx.$$

Hence

$$y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}}\{x(\beta+1)-y\alpha\}\partial_{\ell,m}(x)dx. \quad (1.13)$$

Now, from the middle term of (1.6), we have

$$-(\ell+m)[U_{\alpha,\beta,\ell,m}(y)-U_{\alpha,\beta,\ell-1,m}(y)]=-(\ell+m)[\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$$

$$-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\{\ \partial_{\ell-1,m}(x)dx\ ] \quad (1.14)$$

$$=-(\ell+m)\left[\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\{\partial_{\ell,m}(x)-\partial_{\ell-1,m}(x)\}dx\right]$$

$$=-(\ell+m)[\ \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\{-\frac{x}{\ell+m}\frac{d}{dx}\partial_{\ell,m}(x)\}dx\ ]$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^\infty \frac{x^{\alpha}}{(x+y)^{\alpha+\beta+1}}d\partial_{\ell,m}(x)dx$$

Intergrating by parts, we see that the above experession becomes

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \left[\left\{ - \frac{x^{\alpha}}{(x+y)^{\alpha+\beta+1}} \partial_{\ell,m}(x) \right\}_0^{\infty} - \right.$$

$$\left. \int_0^{\infty} \frac{(x+y)^{\alpha+\beta+1} \alpha x^{\alpha-1} - x^{\alpha}(\alpha+\beta+1)(x+y)^{\alpha+\beta}}{(x+y)^{2(\alpha+\beta+1)}} \partial_{\ell,m}(x)dx \right]$$

$$= -\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^{\infty} \frac{x^{\alpha\ 1}\{\alpha(x+y) - x(\alpha+\beta+1)\}}{(x+y)^{\alpha+\beta+2}} \partial_{\ell,m}(x)dx$$

$$= -\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^{\infty} \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}} \{\alpha y - x(\beta+1)\} \partial_{\ell,m}(x)dx$$

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^{\infty} \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}} \{x(\beta+1) - \alpha y\} \partial_{\ell,m}(x)dx$$

the integrated term vanishing, since (1.10) holds. Since right hand side of (1.13) and (1.14) are the same, therefore the first equality of (1.6) follows.

By the first equation of (1.6)

$$y\frac{d}{dy} U_{\alpha,\beta,\ell,m}(y) = -(\ell+m)[U_{\alpha,\beta,\ell,m}(y) - U_{\alpha,\beta,\ell\ 1,m}(y)]$$

$$= -\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^{\infty} \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}} \{-x\frac{d}{dx} \partial_{\ell,m}(x).\} \mathrm{dx}$$

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^{\infty} \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}} \tau_{\ell,m}(x)dx.$$

$$= B_{\alpha,\beta,\ell,m}(y)$$

This proves the second equalityof (1.6)

2.

For any number p, used an index (exponent), and such that $p > 1$, we write $q = \frac{p}{p-1}$, so that p and q are conjugate indices in the sense of Holdr's inequality, i.e. $\frac{1}{p}+\frac{1}{q}=1$. Morevoer, we define q to be if $p = 1$. We use $\{d,e,f\ldots\}$ to denote a positive constant depending only on the parameters $d,e,f,\ldots$ not necessarily on the two occurrences. Inequality of the form $L \leq c(d,e,f,\ldots.)R$ are to be interpreted as meaning "if the expression R is finite, then the expression L is finite and satisfies the inequality."

3. Known Results

The following results we due to Mishra and Singh [6].

Theorem A. If $\ell > n \geq 1, \alpha > 0,\ \beta > -1, m > -1, then$

$$s(x) \rightarrow (D,\alpha,\beta)(C,\ell-1,m),$$

whenever $s(x) \rightarrow s(D,\alpha,\beta)(C,n-1,m)$.

Theorem B. Let $\ell > n \geq 0, r \geq 0$ and m > -1. Suppose that s (x) is summale (C,r,m) to s and that $\int_1^\infty \frac{\partial_{n,m}(x)}{x^{\beta+2}}dx$ converges, then $s(x)$ is summable $(D,\alpha,\beta)(C,\ell,m)$ to S.

4. DEFINITION OF STRONG SUMMABILITY BASED ON $(D,\alpha,\beta)(C,\ell,m)$ SUMMABILITY METHODS

Suppose that $\alpha > 0, \beta > -1, \ell \geq, m \geq -1$ and $p \geq 1$, if

$$\int_0^x \left|U_{\alpha,\beta,\ell-1,m}(y)-s\right|^p dy = o(x) a\ n\ d\ x \rightarrow \infty \tag{4.1}$$

Then we say that the function s(x) summable $[(D,\alpha,\beta)(C,\ell,m]_p$ or strongly summable $(D,\alpha,\beta)(C,\ell,m)$ with index p, to the sum s, if

$$\int_0^x \left|U_{\alpha,\beta,\ell-1m}(y)\right|^p dy = O(x) \text{ as } x \rightarrow 0 \tag{4.2}$$

then we say that the function s (x) is bounded $[(D,\alpha,\beta)(C,\ell,m)]_p$ or strongly bounded $(D,\alpha,\beta)(C,\ell,m_p$ with index p.

When $p = 1$, $[(D,\alpha,\beta)(C,\ell,m)]_p$ and $[D,\alpha,\beta)(C,\ell,m)]$ denote the same method. The summability $(D,\alpha,\beta)(C,\ell-1,m)$ may be regarded as the case $p=\infty$ of summability $[(D,\alpha,\beta)(C,\ell,m)]_p$ as is evident by the following equality

$$\lim_{p\to\infty}\{x^{-1}\int_0^x \left|U_{\alpha,\beta,\ell,m}(y)-s\right|^p dy\}^{1/p} = \sup \left|U_{\alpha,\beta,\ell-1,m}(y)-s\right|^p .$$

We many also regard boundendness $(D,\alpha,\beta)(C,\ell-1,m)$ as the case $p = \infty$ of boundedness $[(D,\alpha,\beta)(C,\ell-1,m)]_p$.

5. SOME ELEMENTARY RESULTS

Theorem 5.1: If the function $s\ (x)$ is summable $[(D,\alpha,\beta)(C,\ell,m)]_p$ to the sum s, where $1 \le p \le \infty, \alpha > 0, \beta > -l, m > -1 \& \ell \ge 1$, then it is summable $(D,\alpha,\beta)(C,\ell,m)$ to the same sum.

In order to prove this Theorem, we require the following Lemma.

Lemma 5.1: Let $\ell > n \ge 1, \alpha > 0, \beta > -1, m > -1$, suppose that $I_{n-1,m}$ converges. Then for a given $x > 0$,

$$(I_{n-1,m} = \int_l^\infty \frac{\partial_{n-1,m}(x)}{x^{\beta+2}} dx)$$

$$U_{\alpha,\beta,l-1,m}(x) = \frac{\Gamma(\ell+m)}{\Gamma(\ell-n)\Gamma(n+m)} x^{-(\ell+m-1)} \int_0^x y^{n+m-1}(x-y)^{\ell-n-1} U_{\alpha,\beta,n-1,m}(y)dy \quad (5.1)$$

where (5.1) is to be taken as including the assertion that both sides exist. This Lemma has been proved by Mishra and Singh [6].

Proof of Theorem 5.1: We observed that, if the function $s(x)$ is summable$[D,\alpha,\beta)(C,\ell,m]_p$ to the sum s, for any p such that $1 < p \le \infty$, then it is summable $[D,\alpha,\beta)(C,\ell,m]$ to s. We may therefore suppose that $p = 1$, i.e.,

$$\int_0^x \left|U_{\alpha,\beta,\ell-1,m}(y)-s\right|dy = o(x)$$

Further, we may suppose that s = 0. Then replacing ℓ by $\ell+1$ and n by n+1 in

Lemma 5.1 we have

$$U_{\alpha,\beta,\ell,m}(x)\frac{\Gamma(\ell+m+1)}{\Gamma(\ell-n)\Gamma(n+m+1)}x^{-(\ell+m)}\int_0^x y^{n+m}(x-y)^{\ell-n-1}U_{\alpha,\beta,n,m}(y)dy \quad (5.2)$$

Taking $n=\ell-1$ in (5.2), we have

$$U_{\alpha,\beta,\ell,m}(x)=(\ell+m)x^{-(\ell+m)}\int_0^x y^{m+\ell-1}U_{\alpha,\beta,\ell-1,m}(y)dy$$

Now

$$|U_{\alpha,\beta,\ell,m}(x)| \le (\ell+m)x^{-(\ell+m)}\int_0^x y^{m+\ell-1}|U_{\alpha,\beta,\ell-1,m}(y)|dy$$

Intergrating by parts, we have

$$|U_{\alpha,\beta,\ell,m}(x)| \le (\ell+m)x^{-(\ell+m)}\left[\{o(x)y^{m+\ell-1}\}_0^x+(m+\ell-1)\int_0^x o(x)y^{m+\ell-2}dy\right]|$$

$$=o(1).$$

Hence $\quad U_{\alpha,\beta,\ell,m}(x)=o(1).$

This proves Theorem 5.1.

Theorem 5.2 : If the funntion $s(x)$ is summable $[(D,\alpha,\beta)(C,n,m)]_p$ to s, where $p>1$, $\ell>n+\dfrac{1}{p}$, $n\ge 1$, then it is summable $(D,\alpha,\beta,)(C,\ell-1,m)$ to the sum

Proof : We may evidently suppose that $s=0$. By Lemma 5.1, we have

$$U_{\alpha,\beta,\ell-1,m}(x)=\frac{\Gamma(\ell+m)}{\Gamma(\ell-n)\Gamma(n+m)}x^{-(\ell+m-1)}\int_0^x y^{n+m-1}(x-y)^{\ell-n-1}U_{\alpha,\beta,n-1,m}(y)dy$$

Now

$$|U_{\alpha,\beta,\ell-1,m}(x)| \le Cx^{-(\ell+m-1)}\int_0^x y^{n+m-1}(x-y)^{\ell-n-1}|U_{\alpha,\beta,n-1,m}(y)|dy$$

$$= Cx^{-(\ell+m-1)}\int_0^x \{y^{\frac{m+n-1}{p}}\ |U_{\alpha,\beta,n-1,m}(y)|\}\{y^{\frac{m+n-1}{q}}(x-y)^{\ell-n-1}\}dy$$

Applying Hoder's inequality with indices p and q, we have

$$|U_{\alpha,\beta,\ell-1,m}(x)| \le Cx^{-(\ell+m-1)}\{\int_0^x y^{n+m-1}\ |U_{\alpha,\beta,n-1,m}(y)|^p dy\}^{1/p}\{\int_0^x y^{n+m-1}(x-y)^{(\ell-n-1)q}dy\}^{1/q}$$

$$= C\{x^{-(\ell+m-1)}o(x)^{\frac{n+m-1}{p}+1/p+\frac{n+m-1}{q}+(\ell-n-1)+1/q}$$

$$= o(x^0) = o(1)$$

Provided that $\ell > n + 1/p$, $n \ge 1$.

Hence $U_{\alpha,\beta,\ell-1,m}(x) = o(1)$ as $x \to \infty$

This proves Theorem 5.2

Theorem 5.3 : Suppose that $\ell \ge 1$. $1 \le p \le q \le \infty (p < \infty); \ell > n + 1/p - 1/q$, where $>$ maybe replaced by $\le$ in the last inequality in the case, $1 < p$, $q < \infty$; or $1 = p, q = \infty$.

Then: (I) $[(D,\alpha,\beta)(C,n,m.]_p \Rightarrow [(D,\alpha,\beta)(C,\ell,m)]_q$.

(II) for any s

$$\sup_x\{x^{-1}\int_0^x |U_{\alpha,\beta,\ell-1,m}(y) - s|^q dy\}^{1/q} \le C(p,q,\ell,m)$$

$$\sup_x\{x^{-1}\int_0^x |U_{\alpha,\beta,n-1,m}(y) - s|^p dy\}^{1/p} \qquad (5.3)$$

The case $p = q = \infty$ is given by Theorem A.

For the proof of Theorem 5.3 we require few Lemmas.

Lemma 5.2 : Let $0 < \mu < 1$,

$$S(w) = \frac{1}{w}\int_0^w \phi(x)dx \qquad (\phi(x) \ge 0)$$

and $s = \sup_{0<w<\infty} S(w)$

Then $\int_0^w x^{-\mu}\phi(x)dx \le c(\mu)sw^{1-\mu}$

If we take, $s\ (w)= o(1)$, as $w \to \infty$, then

$$\int_0^w x^{-\mu}\phi(x)dx = o(w)^{1-\mu}$$

This Lemma is given by Mishra [4].

Lemma 5.3: If $f(x) \ge 0$ and $f(x) \in L^p(0,y), p > 1,$ and if

$$f_a(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,$$

where $0 < \alpha < \frac{1}{p}$ and $q = \frac{p}{1-p\alpha,}$ then $f_\alpha(x) \in L^q(o,y)$

and $\{\int_0^y \{f_\alpha(x)\}^q dx\}^{1/q} \le C\{\int_0^y \{f(x)\}^p dx\}^{1/p},$

where $C = C(p,\alpha) = C(p,q)$.

This Lemma has been proved by Hady, Little wood and Polya [2].

We pass now to the proof of Theorem 5.3, we may evidently suppose that s = 0. Since Theorem 5.3 (I) is a corollary of Theorem 5.3 (II), it is sufficient ot prove Theorem 5.3 (II) Let $\ell = n + \gamma$

Case (i) : $1 \le p \le q \le \infty$ and $\gamma > \frac{1}{p} - \frac{1}{q},$

Case (ii) : $1 < p < q < \infty$ and $\gamma = \frac{1}{p} - \frac{1}{q},$

Case (iiia): $1 < p < q = \infty$ and $\gamma > \frac{1}{p} - \frac{1}{q},$

Case (iiib) $1 = p < q = \infty$ and $\gamma \geq \frac{1}{p} - \frac{1}{q}$.

Proof of Case (i) Let S denote the superemum on the right of (5.3) suppose that $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$ and $0 < \delta < 1$, so that $r(\ell - n - 1) > -1$ $r \geq 1$.

By Lemma 5.1, We have

$$\left|U_{\alpha,\beta,\ell-1,m}(x)\right| \leq Cx^{-(\ell+m-1)} \int_0^x y^{n+m-1} (x-y)^{\ell-n-1} \left|U_{\alpha,\beta,n-1,m}(y)\right| dy$$

$$= Cx^{-(\ell+m-1)} \int_0^x \{(x-y)^{r(\ell-n-1)(1-\frac{1}{p})} y^{r(n+m-1+\delta)(1-\frac{1}{p})-\delta(1-\frac{1}{p})}\}\{y^{-\delta(\frac{1}{p}-\frac{1}{q})} \left| U_{\alpha,\beta,n-1,m}(y)\right|^{p(\frac{1}{p}-\frac{1}{q})}\}$$

$$\{x-y)^{r(\ell-n-1)\frac{1}{q}} y^{r(n+m-1+\delta)\frac{1}{q}-\frac{\delta}{q}} \left|U_{\alpha,\beta,n-1,m}(y)\right|^{p/q} dy\}$$

Applying Holder's inequality with indices $\frac{p}{p-1}, \frac{pq}{q-p}$ and q. we have

$$\left|U_{\alpha,\beta,\ell-1,m}(x)\right| \leq Cx^{-(\ell+m-1)} \{\int_0^x (x-y)^{r(\ell-n-1)} y^{r(n+m-1+\delta)-\delta} dy\}^{1-\frac{1}{r}} \{\int_0^x y^{-\delta} \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{(\frac{1}{p}-\frac{1}{q})}$$

$$\{\int_0^x (x-y)^{r(\ell-n-1)} y^{r(n+m-1+\delta)-\delta} \left| U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/q}$$

$$= Cx^{-(\ell+m-1)} x^{\{r(\ell-n-1)+r(n+m-1+\delta)-\delta+1\}(1-\frac{1}{p})} x^{(-\delta+1)(\frac{1}{p}-\frac{1}{q})} S^{p(\frac{1}{p}-\frac{1}{q})} \{\int_0^x (x-y)^{r(\ell-n-1} y^{r(n+m-1+\delta)-\delta}$$

$$\left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/q} \text{ (by lemma 5.2)} \tag{5.4}$$

Hence

$$\left|U_{\alpha,\beta,\ell-1,m}(x)\right|^q \leq Cx^{-\{r(\ell+m-2+\delta)-\delta+1\}} S^{q-p} \{\int_0^x (x-y)^{r(\ell-n-1)} y^{r(n+m-1+\delta)-\delta} \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}$$

$$\int_0^w \left| U_{\alpha,\beta,\ell-1,m}(x)\right|^q dx \leq CS^{q-p} \int_0^w \left[x^{-\{r(\ell+m-2+\delta)-\delta+1\}} \int_0^x (x-y)^{r(\ell-n-1)} y^{r(m-1+\delta)-\delta} \left|U_{\alpha,\beta,n-1,m}{}^{(y)}\right|^p dy\right]dx$$

Interchanging the order of integration, we have

$$\int_0^w | U_{\alpha,\beta,\ell-1,m}(x)|^q dx \le CS^{q-p} w^\delta \int_0^w (y^{r(n+m-1+\delta)-\delta} | U_{\alpha,\beta,n-1,m}(y)|^p dy \int_y^w (x-y)^{r(\ell-n-1)} (x)^{-r(\ell+m-2+\delta)-1} dx$$

$$\le CS^{q-p} w^\delta \int_0^w (y^{r(n+m-1+\delta)-\delta} |U_{\alpha,\beta,n-1,m}(y)|^p dy . y^{-r(n+m-1+\delta}$$

$$\le CS^{q-p} w^\delta \int_0^w y^{-\delta} |U_{\alpha,\beta,n-1,m}(y) / dy,$$

$$\le CS^{q-p} w^\delta S^p w^{1-\delta}, \text{ by Lemma. 5.2}$$

$$\le CS^q w$$

Since

$$\int_u^w (y-u)^n y^{-n-\ell-1} dy \le \int_u^\infty (y-u)^n y^{-n-\ell-1} dy$$

$$= \frac{u^{-\ell}\Gamma(n+1)\Gamma(\ell)}{\Gamma(\ell+n+1)}, \qquad (n > -1, u > 0, \ell > 0).$$

Since n+m-1>-1, we may choose δ such that r (n+m-1+δ)>0.

Thus by using Lemma 5.2, we have.

$$\sup_{w\ge 0}\{w^{-1}\int_0^w |U_{\alpha,\beta,\ell-1,m}(x)|^q dx\}^{1/q} \le C(p,q,n,m) \sup_{w\ge 0}\{w^{-1}\int_0^w |U_{\alpha,\beta,n-1,m}(y)|^p dy\}^{1/p}$$

When $p = 1$, The first factor of (5.4) is to be omitted. Similarly when $q = p$, the second brackted factor in (5.4) is to be omitted.

This proves Case (i)

Proof of Case (ii)

$$1 < p < q < \infty, \qquad 0 < \gamma = \frac{1}{p} - \frac{1}{q} < \frac{1}{p}$$

By Lemma 5.1, we have

$$\left|U_{\alpha,\beta,l-1,m}(x)\right| \leq Cx^{-(n+\gamma+m-1)}\int_0^x y^{n+m-1}(x-y)^{\gamma-1}\left|U_{\alpha,\beta,n-1,m}(y)/dy\right.$$

$$= Cx^{-n-\gamma-m+1}\int_0^{x/2} y^{n+m-1}(x-y)^{\gamma-1}\left|U_{\alpha,\beta,n-1,m}(y)\right|dy + Cx^{-n-\gamma-m+1}\int_{x/2}^x (x-y)^{\gamma-1}y^{n+m-1}\left|U_{\alpha,\beta,n-1,m}(y)\right|dy$$

$= C(I_1 + I_2)$, say.

Hence, by Minkowski's inequality, it is sufficient to prove that

$$\sup_x\{x^{-1}\int_0^x |I_1|^q dy\}^{1/q} \leq C\sup_x\{x^{-1}\int_0^x \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/p} \tag{5.5}$$

and

$$\sup_x\{x^{-1}\int_0^x |I_2|^q dy\}^{1/q} \leq C\sup_x\{x^{-1}\int_0^x \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/p} \tag{5.6}$$

Proof of (5.5) :

Since $0 < \gamma < 1, (x-y)^{\gamma-1}$ Increases in (0,x/2), therefore we have

$$I_1 \leq Cx^{-n-\gamma-m+1+\gamma-1}\int_0^{x/2} y^{n+m-1}\left|U_{\alpha,\beta,n-1,m}(y)\right|dy.$$

Proceeding as in the case (i), we see that (5.5) is established.

Proof of (5.6) :

$$I_2 \leq C\int_{x/2}^x (x-y)^{\gamma-1}y^{-\gamma}\left|U_{\alpha,\beta,n-1,m}(y)\right|dy.$$

Applying Lemma 5.3, we have

$$\{\int_0^x |I_2|^q dy\}^{1/q} \leq C\{\int_0^x y^{(1-\frac{p}{q})}\left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/p}$$

Since $1 - p/q < 1$, therefore we can apply Lemma 5.2. Thus.

$$\sup_x\{x^{-1}\int_0^x \left|U_{\alpha,\beta,l-1,m}(y)\right|^q dy\}^{1/q} \leq C(p,q,n)\sup_x\{x^{-1}\int_0^x \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/p}$$

This proves the Case (ii)

Proof of Case (iiia) is given by Theorem5.2 Case (iiib) is given by Theorem 5.1. Thus, we have completely proved Theorem 5.3.

Theorem 5.4: The necessary and sufficient conditions for the function s (x) to be summable $[(D,\alpha,\beta])(C,l,m)]_p$ to s are that it be summable $[(D,\alpha,\beta])(C,l,m)]_p$ to s and

$$\int_0^x \left| y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y)\right|^p dy = o(x) \text{ as } x \to \infty. \qquad (\ell > 1, m > -1).$$

Proof. This result is analogous to Hyslop's Theorem 3 [1] on strong Cesaro summability. Since summability $[(D,\alpha,\beta])(C,l,m)]_p$ implies summability $[(D,\alpha,\beta])(C,l,m)]$ it follows from (1.7) and Minkowski's inequality, that since $U_{\alpha,\beta,\ell,m}(y) - s = o(1)$ as y $\to \infty$.

$$\{\int_0^x \left|U_{\alpha,\beta,\ell-1}(y) - s\right|^p dy\}^{1/p} - \{\int_0^x \left|U_{\alpha,\beta,\ell,m}(y) - s\right|^p dy\}^{1/p} \le \frac{1}{|\ell+m|}\{\int_0^x \left| y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y)\right|^p dy\}^{1/p} \qquad (5.7)$$

$$\le \{\int_0^x \left|U_{\alpha,\beta,\ell-1,m}(y) - s / ^p dy\}^{1/p} + \{\int_0^x \left|U_{\alpha,\beta,\ell,m}(y) - s\right|^p dy\}^{1/p}$$

$$= o(x)^{1/p} + o(x)^{1/p}$$

$$= o(x)^{1/p}$$

This implies that $\int_0^x \left| y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y)\right|^p dy = o(x)$ as $x \to \infty$

Conversely suppose that function s (x) is summable $[(D,\alpha,\beta])(C,l,m)]$ to s and

$$\int_0^x \left| y\frac{d}{dy}U_{\alpha,\beta,l,m}(y)\right|^p dy = d\,(x) \text{ as } x \to \infty$$

Again using (5.7), we have

$$\int_0^x \left|U_{\alpha,\beta,\ell-1,m}(y)-s\right|^p dy = o(x) \quad \text{as } x \to \infty$$

This completes the proof of Theorem 5.4.

6. RELATIVE STRENGTH BETWEEN $[D,\alpha,\beta)(C,\ell,m]_p$ AND $[C,n,m]_p$

Theorem 6.1: Suppose that $1 \le p \le \infty$ and $n \ge 0$. If the function $s\ (x)$ is summable $[C,n,m]_p$ to the sum s, it summable $[D,\alpha,\beta)(C,\ell,m]_t$ to the sum s for every $t \ge 1$, $\ell \ge 1$.. Wherever $(D,\alpha,\beta)\ (C,\ell-1,m)$ method is appliable.

The proof of this theorem follows immediately from B and the chain of implications

$[C,n,m]_p \Rightarrow (C,n.m) \Rightarrow (D,\alpha,\beta)(C,\ell-1,m) \Rightarrow [(D,\alpha,\beta)(C,\ell,m]_t$. for every $t = \ge 1$,

Theorem 6.2 Suppose $1 \le p \le \infty$ and $\ell \ge 1, m > -1, \ell > \delta > 0$. If the function s (x) is bounded $[(D,\alpha,\beta,)(C,\ell-\delta,m]_p$ and summable $[D,\alpha,\beta,)(C,\ell-\delta+1,m]_p$ to the sum s, it is summable $[D,\alpha,\beta,)(C,\ell,m]_p$ to s

Proof. We may suppose that $s = 0$, $0 < \delta < 1/p$. By Lemma 5.1 replacing n by $\ell-\delta$ we have

$$U_{\alpha,\beta,\ell-1,m}(x) = Cx^{-(\ell+m-1)} \int_0^x (x-y)^{\ell-\ell+\delta-1} y^{\ell-\delta+m-1} U_{\alpha,\beta,\ell-\delta-1,m}(y)dy$$

$$= Cx^{-(\ell+m-1)} \int_0^{x(1-p)} (x-y)^{\delta-1} y^{\ell-\delta+m-1} U_{\alpha,\beta,\ell-\delta-1,m}(y)dy + Cx^{-(\ell+m-1)} \int_{x(1-\delta)}^x (x-y)^{\delta-1} y^{\ell-\delta+m-1} U_{\alpha,\beta,\ell-\delta-1,m}(y)dy.$$

$= I_1(x) + I_2(x)$, say,

ρ being a fixed number such that $0 < \rho < 1/2$. Let now

$$S = \sup_x \{x^{-1} \int_0^x \left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^p dy\}^{1/p}. \tag{6.1}$$

We have $\left|I_2(x)\right|^p \le C\{x^{-(\ell+m-1)} \int_{x(1-\rho)}^x (x-y)^{\delta-1} y^{\ell-\delta+m-1} \left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right| dy\}^p$

$$\leq C\{\int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{-\delta}\left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|dy\}^{p}$$

or

$$\left|I_2(x)\right|^{p}\leq C\{\int_{x(1-\rho)}^{x} (x-y)^{\frac{\delta-1}{p}} y^{-\delta}(x-y)^{\frac{\delta-1}{q}}\left|U_{\alpha,\beta,\ell-1,m}(y)\right|dy\}^{p}$$

Applying Holder's inequality with indices p and q, we have

$$\left|I_2(x)\right|^{p}\leq C\{\int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{-\delta p}\left|U_{\alpha,\beta,\ell-1,m}(y)\right|^{p}dy\}\{\int_{x(1-\rho)}^{x} (x-y)^{\delta-1}dy\}^{p/q}$$

$$=C(\ell,m,\delta,p)x^{\delta(p-1)}\rho^{\delta(p-1)}-\int_0^t \{x^{\delta(p-1)}\int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{-\delta p}\left|U_{\alpha,\beta,\ell-1,m}(y)\right|^{p}dy.$$

Hence

$$\int_0^t \left|I_2(x)\right|^{p}dx\leq C\rho^{\delta(p-1)}-\int_0^t \{x^{\delta(p-1)}\int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{-\delta p}\left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^{p}dy\}dx$$

$$\leq Ct^{\delta(p-1)}\rho^{\delta(p-1)}\int_0^t dx\int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{-\delta p}\left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^{p}dy.$$

Interchanging the order of integration, we have

$$\int_0^t \left|I_2(x)\right|^{p}dx\leq Ct^{\delta(p-1)}\rho^{\delta(p-1)}\int_0^t y^{-\delta p}\left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^{p}dy\int_{y}^{y\,1-\rho} (x-y)^{\delta-1}dx$$

$$=Ct^{\delta p}\rho^{\delta p}(1-\rho)^{\delta}\int_0^t y^{-\delta p}\left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^{p}dy$$

Applying Lemma 5.2 we have

$$\int_0^t \left|I_2(x)\right|^{p}dx\leq CS^{p}\rho^{\delta p}(1-\rho)^{-\delta}t,$$

(since $\delta p<1$), the constant being independent of ρ. Replacing ℓ by $n+1$ in Lemma 5.1, we have

$$U_{\alpha,\beta,n,m}(x) = Cx^{-(n+m)}\int_0^x y^{n+m-1}(x-y)^{n+1-n-1}U_{\alpha,\beta,n-1,m}(y)dy$$

or

$$U_{\alpha,\beta,n,m}(x) = Cx^{-(n+m)}\int_0^x y^{n+m-1}U_{\alpha,\beta,n-1,m}(y)dy$$

we have

$$I_1(x) = Cx^{-(\ell+m-1)}\int_0^{x(1-\rho)} (x-y)^{\delta-1}y^{\ell-\delta+m-1}U_{\alpha,\beta,\ell-\delta-1,m}(y)dy$$

Integrating by parts

$$I_1(x) = Cx^{-(\ell+m-1)}[(x-y)^{\delta-1}y^{\ell-\delta+m}U_{\alpha,\beta,\ell-\delta,m}(y)]_0^{x(1-\rho)} + Cx^{-(\ell+m-1)}\int_0^{x(1-\rho)} (\delta-1)(x-y)^{\delta-2}y^{\ell-\delta+m}U_{\alpha,\beta,\ell-\delta,m}(y)dy$$

$$= Cx^{-(\ell+m-1)}[(\rho x)^{\delta-1}x^{\ell-\delta+m}(1-\rho)^{\ell-\delta+m}U_{\alpha,\beta,\ell-\delta,m}\{x(1-\rho)\}] + Cx^{-(\ell+m-1)}(\delta-1)\int_0^{x(1-\rho)} (x-y)^{\delta-2}y^{\ell-\delta+m}U_{\alpha,\beta,\ell-\delta,m}(y)dy$$

$$= C(1-\rho)^{\ell-\delta+m}\rho^{\delta-1}U_{\alpha,\beta,\ell-\delta,m}\{x(1-\rho)\} + Cx^{-(\ell+m-1)}\int_0^{x(1-\rho)} (x-y)^{\delta-2}y^{\ell-\delta+m}U_{\alpha,\beta,\ell-\delta,m}(y)dy$$

$$= I_{1.1}(x) + I_{1.2}(x), (say).$$

$$I_{1.1}(x) = C\rho^{\delta-1}(1-\rho)^{\ell-\delta+m}U_{\alpha,\beta,\ell-\delta,m}\{x(1-\rho)\}$$

$$|\ I_{1.1}(x)|^p \le C\rho^{\delta p-p}(1-\rho)^{p(\ell-\delta+m)}|\ U_{\alpha,\beta,\ell-\delta,m}\{x(1-\rho)\}|^p$$

Integration both sidess, we get

$$\int_0^t |I_{1.1}(x)|^p dx \le C(\ell,m,\delta,\rho,p)\int_0^t |U_{\alpha,\beta,\ell-\delta,m}\{x(I-\rho)\}|^p dx \tag{6.2}$$

$$= o\ (t).$$

$$|I_{1.2}(x)|^p = C|x^{-(\ell+m-1)}\int_0^{x(1-\rho)} (x-y)^{\delta-2}y^{\ell-\delta+m}U_{\alpha,\beta,\ell-\delta,m}(y)dy|^p$$

if $p > 1$ we have, by Holder's inequality with indices p and q,

$$\left|I_{1,2}(x)\right|^p \le Cx^{-(\ell+m-1)p}\{\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\}\{\int_0^{x(1-\rho)} \{(x-y)^{\delta-2}\}^q dy\}^{p/q}$$

$$\left|I_{1,2}(x)\right|^p \le Cx^{-(\ell+m-1)p}\{\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\Big[\{\frac{-(x-y)^{(\delta-2)q+1}}{(\delta-2)q+1}\Big]_0^{x(1-\rho)}\}^{p/q}$$

$$\le Cx^{-(\ell+m-1)p}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\left\{\frac{-[(\rho x)^{(\delta-2)q+1}-x^{(\delta-2)q+1}]^{p/q}}{[(\delta-2)q+1]^{p/q}}\right\}$$

$$\le Cx^{-(\ell+m-1)p}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\left\{\frac{x^{\{(\delta-2)q+1\}p/q}[\rho^{(\delta-2)q+1}-1]^{p/q}}{[(\delta-2)q+1]^{p/q}}\right\}$$

$$\le Cx^{-(\ell+m-1)p}\int_0^{x(1-p)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\,x^{\delta p-p-1} \quad (\text{since } 1/p+1/q=1)$$

$$\le Cx^{-(\ell+m-1)p+\delta p-p-1}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy$$

$$\le Cx^{-\ell p-mp+\delta p-1}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy$$

$$\le Cx^{-1}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p dy$$

$$== O(x^{-1})o(x) = o(1).$$

If $p = 1$, we have

$$I_{1,2}(x)\Big| \le O(x^{-\ell-m+\delta-1}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right| y^{\ell-s+m}\,dy.$$

$$\le O(x^{-\ell-m+\delta-1})o(x^{\ell-\delta+m+1})$$

$$= o(1),$$

and so in either case $\int_0^t \left| I_{1,2}(x)\right|^p dx \le C\int_0^t o(1)dx. = o(t)$ (6.3)

It follows by Minkowski's inequality that

$$\limsup \left\{t^{-1}\int_0^t \left|U_{\alpha,\beta,\ell-1,m}(x)^p\right|dx\right\}^{1/p}$$

$$\le SC(\ell,m,\delta,p)\rho^{\delta}(1-\rho)^{-\delta/p}+o(1)+o(1)$$

$$\le SC(\ell,m,\delta,p)\rho^{\delta}(1-\rho)^{-\delta/p} \qquad (6.4)$$

Making $\rho \to 0$, it follows now that limit sup on the left of (6.4) is 0, and this complets the proof of Theorem. 6.2.

Theorem 6.3: Suppose that $1 < p \le \infty, \alpha > 0, \beta \ge -1, m > -1, \ell > 1$ and (1.5) holds. If the function s (x) is summable $[D,\alpha,\beta)(C,\ell,m]_p$ to the sum s, then the function x a (x) is summable $[(D,\alpha,\beta)(C,\ell+1,m)]_p$ to the sum 0.

Theorem 6.4 :Suppose that $1 \le p \le \infty, \alpha > 0, \beta \ge -1,$ and (1.5) holds. If the function $s(x)$ is summable $[(D,\alpha,\beta)(C,\ell+1,m]_p$ to s and xa(x) is summable $[(D,\alpha,\beta)(C,\ell+1,m)]_p$ to the sum 0, then the function $s(x)$ is summable $[(D,\alpha,\beta)(C,\ell,m)]_p$

Theorem 6.5: Suppose that $1 \le p \le \infty, \alpha > 0, \beta \ge -1, \delta > 0$ and (1.5) holds If the function s (x) is summable $[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$ to s and the function $xa(x)$ is bounded $[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$ then the function $s(x)$ is summable $[(D,\alpha,\beta)(C,1-\delta,m)]_p$ to the sum s.

Proof of Theorem 6.3 follows immediately Theorem 5.4 while Theorem 5.4 follows directly from (1.6) and Minkowski's inequality.

To prove Theorem 6.5, we first note that if the function $s(x)$ is summable $[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$ then the function $xa(x)$ is summable $[(D,\alpha,1\beta)(C,\ell-\delta+1,m)]_p$. If also the function $xa(x)$ is bounded $[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$, then it is summable $[(D,\alpha,\beta)(C,\ell,m)]_p$ to 0, and since summability$[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$ implies summability $[(D,\alpha,\beta)(C,\ell,m)]_p$, Theorem 6.4 completes the proof of Theorem 6.5

REFERENCES

1. J.M. Hyslop: Note on the strong summability of series. Proc. Glasgow Math. Assoc. 1, (1951/53), 16-20.
2. G.H. Hardy, J.E. Littlewood, and G. Polya : Inequalities. (Cambridge, 1959), P. 290.
3. B. Kwee : On Generalized translated quasi-cesaro summability. Pacific Journal of Mathematics 36, (No. 3) (1971) 731-740.
4. B.P. Mishra. : Strong summability of infinite series on a scale of Abel type summability methods. Proc. Cambridge Philos. Soc. 63 (1967) 119-127
5. B.P. Mishra & A.P. Srivastava: some remarks on absolute summability of function based on (C,α,β) summability methods, Nat. Acad. of Mathermaticcs 13 (1999).
6. B.P. Mishra. & S.K. Singh: On product summability methods ($(D,\alpha,\beta)(C,\ell,m)$ of function communicated for publication,
7. Singh, B.N. : Some remarks on summability methods function, Ph. D. Theses, Gorakhpur University (1992).

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 77-88

EFFECT OF POLARIZABILITY ON NUCLEATION PHENOMENON DURING ICE GLACIATION DUE TO EXTERNAL ELECTRIC FIELD

N.SINGH*, DEVENDRA SINGH, VIKAS MISHRA** AND PRAKASH MISHRA****

(Received 14-08-2004)

ABSTRACT

The polarizability of water vapour molecules in presence of external electric field plays an important role in the nucleation rate of water vapour condensation and ice glaciation. The polarizability decreases with increase in the temperature and hence there is decrease in Gibb's free energy and hence nucleation rate is decreased .Thus polarizability controls the ice glaciation in clouds.

Keywords and phrases: Polarization. Nucleation rate ,Gibb's free energy, Critical nuclei.

INTRODUCTION

The polarizability plays a dominant role in inducing the electric dipole on water vapour molecules. The contribution to polarization of water vapour molecules coming from the electric field generated by the droplet dipole, which increases the interaction between droplet and water vapour molecules. This would enhance the effect described by Murino [18], but the interaction due to this additional factor would not be steady and hence would produce only transient effect.

Ice nucleation is a fundamental cloud process. The process involved in ice formation is understood only in the case of homogeneous freezing of supercooled droplets. In contrast, ice formation caused by heterogeneous freezing nuclei at low temperatures are poorly understood. Kärcher and Lohmann [12] considered theoretically the nucleation and initial growth of ice crystals in cerrus clouds at

---

*Department of Physics, Nehru college, Chhibramau -209721, U.P.
**Department of Physics, D.B.S. College, Kanpur-208 006, U.P.

low temperatures (< 235K) prevailing in the upper troposphere and in the tropopause region.

Ice crystals are produced by electric fields in cloud chambers [8] caused by fragmentation of dendritic crystals. The nucleation of ice in supercooled water by electric fields has been demonstrated. Nucleation in the thin liquid filaments, which form during disruption of drops or during movement of drops on surfaces was suggested by Loeb [13] and by Abbas and Latham [1] as a possible mechanism.

Nucleation as a result of distortion of drops by electric forces was reported [1,20,21,28]. When charged, aerosols and other surfaces coming in contact with drops were found to cause nucleation more readily than if no electric charge were present [9,21,23]. The ice formation in the atmosphere is a heterogeneous process initiated by ice forming nuclei [15].

Dufour [7] suggested that freezing could be initiated in supercooled water by means of an electric field. Unfortunately he gave only an insufficient description of his experiment so that no firm conclusions can be drawn. Dufour [7], Rau [22], Salt [25], Gabarashvili and Gliki [9] proposed that electro-freezing of supercooled water is due to dielectric polarization of water in high electric fields.

Collins [4] inferred that the relaxation time is independent of the free energy of fonnation of the nucleus, but it varies as the square of the radius of the critical nucleus. Bartlett et al. [3], as well as Maybank and Barthakur [16] reported the tendency of generation of a few ice splinters during ice crystal growth in strong electric fields. Doolittle and Vali [6] pointed out the effect of an electric field on samples of water at different time temperature cycles.

The formulation of relaxation time required for the attainment of the steady state concentrations of the embryos of the critical size has been discussed [4,11,19,29]. The scavenging of aerosol particles by hydro-meteors under external electric field has been discussed by Wang [30]. Sharma et al. [26] reported that the small values of electric field is equivalent to very high supersaturation ratio to get a nucleus of given size under similar conditions of temperature.

Singh et al. [27] have shown that in the resultant electric effect on a droplet due to an external electric field and the field induced due to the central dipole, the role of nucleation in water vapour condensation and ice glaciation is about hundred times more near breakdown for dry air as compared to that in absence of electric field. Thunderstorm electric field promote droplet growth.

Electric field near triggered lightning channels have been measured with Pockels sensors. Mikki et al. [17] at the International Centre for Lightning Research and Testing at Camp Blanding, Florida. The vertical electric field pulse peaks are in the range from 176 kV/m to 1.5 MV/m (the median is 577 kV/m) and horizontal electric field pulse peaks are in the range from 495 kV/m to 1.2 MV/m (the median is 821 kV/m). The vertical electric field measured very close to the lightning channel tends to increase with an'increase in the previous no current interval.

The atmospheric electric field and the electrical conductivity varies from place to place depending upon certain environmental factors. Deshpande and Kamra [5] gave the surface measurements of the atmospheric electric potential gradient and conductivity at Indian station, Maitri (70^0 45' 52" S, 11^{0}44' 3" E, 117 m above mean sea level), Antarctica, from January 10 to February 24, 1997. The diurnal variation of potential gradient averaged for 20 fair weather days is single periodic with maximum at 1300 UT and another secondary but distinct maximum at 1900 UT and a minimum at 0100 UT. The mean value or the potential gradient is 83 V/m and the maximum and minimum values are 1.55 and 0.73 times the mean value, respectively. The total conductivity is 2.1×10^{-14} S/m and does not show any significant diurnal variation. The diurnal variation of the potential gradient observed at Maitri is similar to the one observed a few days later in the southern Indian Ocean and to the one reported by Kamra et al. [10] with a few hours difference in phase in the northern Indian Ocean.

Rust and Trapp [24] presented the first known vertical profiles of electric field in six winter nimbo stratus clouds in the U.S.A. The maximum magnitude of the vertical component of electric field in the profiles ranged from 1 to 12 kV/m, the maximum horizontal component ranged from 0.2 to 28 kV/m.

Marshall and Stolzenburg [14] used a one-dimensional model to calculate the electrostatic energy of several horizontally the extensive, electrified clouds. Model estimates of the model calculation are initiated with known charge distributions. Small reductions in the charge density of a region often can provide a relatively large amount of energy to derive a flash. Model calculations suggest that positive cloud-to-ground flashes that produce Q-bursts have energies of about 1×10^{10} J. Total electrostatic energy stored in two stratiform clouds of mesoscale convective systems were 5×10^{11} J and 2×10^{12} J. These energies are sufficient to support hundreds or thousands of typical lightning flashes, but only 10-100 of the energetic positive cloud-to-ground flashes with associated Q-bursts.

To authors knowledge the role of effective polarizability has not been considered affecting nucleation rate of ice particles. In present study, we have evaluated the effect of polarizability on nucleation effective polarizability. We estimated nucleation rate of ice particles in external electric field using modified value of polarizability at different temperatures. Due to the effective polarizability α_{eff}, the Gibb's free energy, nucleation rate and radius of nucleus of ice are found to increase.

THEORETICAL CONSIDERATION

The effective or resultant polarizability may be written as

$$\alpha_{eff} = \alpha + \frac{p_0^2}{3kT} \tag{1}$$

Thus, effective polarizability varies inversely as absolute temperature of medium.

In an external electric field, the moment induced on a droplet is

$$\vec{M} = \vec{E}r_i^3 \tag{2}$$

where, r_i is the radius of ice embryo (assumed spherical) and $\vec{E}$, the electric field.

The moment induced on water vapour molecule, present in the cloud is

$$\vec{M}' = \alpha\vec{E} \tag{3}$$

where, α is the polarizability. For α we assume the value 5×10^{-23} cm^3, observing that the contribution from the permanent electric dipole to polarizability is

$$\frac{p_0^2}{3kT} = 2.6\times10^{-23}\ \text{cm}^3$$

where, p is the permanent dipole moment; k, the Boltzmann constant and T, the absolute temperature.

HOMOGENEOUS NUCLEATION

In the classical models of drop condensation and ice nucleation the Gibbs function has been taken into consideration. However, Abraham[2] has shown that the Helmholtz free energy is the proper thermodynamic potential and Gibbs function is only its approximation.

The Gibb's free energy for formation of a ice embryo is given by

$$\Delta G_i = -\frac{4}{3}\pi r_i^3 \Delta G_v + 4\pi r_i^2 \sigma_{i/v} \tag{4}$$

where $\Delta G_v = \dfrac{\rho_i RT \ln S_{v,i}}{M_w}$

R, the universal Gas constant; $S_{v,i}$, the supersaturation ratio of water vapour over ice surface; M_w, the molecular weight of water.

Critical radius of ice nucleus is given by

$$\frac{\partial \Delta G_i}{\partial r_i} = 0$$

The radius of critical ice nucleus is

$$r_i^* = 2\sigma_{i/v} M_w / \rho_i RT \ln S_{v,i} \tag{5}$$

The number ice molecules in a critical nucleus is given by

$$n_i^* = 4\pi r_i^{*3} \rho_i N / 3M_w \tag{6}$$

where, ρ is density of ice; N, the Avogadro number.

The energy of formation of a critical nucleus is

$$\Delta G_i^* = 4\pi r_i^{*2} \sigma_{i/v} / 3 \tag{7}$$

The nucleation rate of ice nuclei is

$$J_i^* = \text{a constt.}\exp\left[-\Delta G_i^* / kT\right] \tag{8}$$

IN PRESENCE OF EXTERNAL ELECTRIC FIELD

Water is strongly polarizable having dipole moment 1.83×10^{-18} esu cm. In an external electric field, the water droplet (or embryo) rapidly increases in size. The expression for the rate of growth of radius of ice embryo has been derived as [27].

$$\frac{dr_i'}{dt} = \frac{\rho_v}{\rho_i}\left[9\alpha_{eff}\lambda E^2 / m_w r_i'\right]^{\frac{1}{2}} \tag{9}$$

where, λ is the mean free path; E, the external electric field; m_w, the mass of water vapour molecule; ρ_v, the density of water vapour and r_i' the radius of ice and embryo in the electric field. Integrating above equation within limits $r_i' = 0$ to $r_i'^*$ (critical radius of ice nucleus) and $t = 0$ to $t = \tau$ (relaxation time)

$$r_i'^* = \left[3\rho_v \tau\left(9\alpha_{eff}\lambda E^2 / m_w\right)^{1/2} / 2\rho_i\right]^{2/3} \tag{10}$$

Analogically, in the external field eqn. (3.32) and (3.33) for the energy of formation and the nucleation rate of ice nucleus, becomes

$$\Delta G_i^{\prime *} = \frac{4\pi}{3} r_i^{\prime *} \sigma_{i/v} \tag{11}$$

and

$$J_i^{\prime} = \text{a constant. } \exp\left(-\frac{\Delta G_i^{\prime *}}{kT}\right) \tag{12}$$

RESULTS

The effective polarizability α_{eff}, of water varies nearly inversely as the absolute temperature. The variation of α_{eff} with temperature has been shown in table 1.

Table 1. : Calculated values of α_{eff} of water molecules with the absolute temperature using $\alpha = 5x10^{-23}$ cm^3.

T(K)	$\alpha_{eff}(x10^{-23})$ cm^3
263	8.006
273	7.896
283	7.794
293	7.698
303	7.609

Using eqn. (9), (10) and (11) the critical radius of the ice nucleus r_i^*, Gibb's free energy of formation ΔG_i^* and natural logarithm of nucleation rate $ln\ J_i^*$ as the function of external electric field at relaxation time 2 x 10^{-6} see at temperatures 263, 273 and 283 K have been shown in Table 2.

Calculated value of critical radius of ice, r_i^*, Gibb's free energy of formation of critical ice nucleus, ΔG_i^*, and natural logarithm of nucleation rate of ice nuclei $ln\ J_i^*$ as the function of the relaxation time at constant electric field 10 esu at temperatures 263, 273 and 283 K are given in Table 3.

Table 2. : Calculated values of $r_i^*, \Delta G_i^*$ and In J_i^* as the function of external electric field at $\tau = 2 \times 10^{-6}$ s and temperatures 263, 173 and 283 K.

E (esu)	$r_i^*(\times 10^{-9})$cm	$\Delta G_i^*(\times 10^{-14})$erg	$\ln J_i^*$
T = 263 K			
5	18.59	14.48	3.99
10	29.50	36.48	10.05
15	38.66	62.63	17.26
T = 273 K			
5	18.51	14.35	3.81
10	29.38	36.16	9.60
15	38.49	62.09	16.48
T = 283 K			
5	18.43	14.23	3.64
10	29.25	35.85	9.18
15	38.33	61.55	15.76

Table 3. : Calculated values of $r_i^*, \Delta G_i^*$ and In J_i^* as the function of relaxation time at E = 10 esu at temperatures 263, 173 and 283 K.

T $(\text{x}10^{-9})$sec	$r_i^*(\times 10^{-9})$cm	$\Delta G_i^*(\times 10^{-14})$erg	$\ln J_i^*$
T = 263 K			
10	86.27	30.92	82.94
20	136.94	78.58	216.52
30	179.44	134.93	371.78
T = 273 K			
10	85.90	30.65	82.06
20	136.35	76.23	206.79
30	178.67	133.77	355.08
T = 283 K			
10	85.52	30.10	78.48
20	135.76	73.91	197.75
30	177.89	132.61	339.56

DISCUSSION

The calculations in Table l, 2 and 3 have been made using the following constants:

ρ_v = 10^{-5} at 10^0 gm cm^{-3}
α = $5.0\mathrm{x}10^{-23}$ cm^3
p = $1.81\mathrm{x}10^{-18}$ esu
k = $1.38\mathrm{x}10^{-16}$ erg K^{-1}
λ = 10^{-5} cm
m_w = $3.0\mathrm{x}10^{-23}$ gm
M_w = 18
ρ_i = 0.917 gm cm^{-3}
$\rho_{i/v}$ = 100 dynes cm^{-1}
R = $8.317\mathrm{x}10^7$ erg K^{-1} mole^{-1}
N_A = $6.025\mathrm{x}10^{23}$

The effective polarizability of water varies approximately inversely as absolute temperature(Table 1). From Table 2 it is clear that radius decreases with increase in temperature. Also Gibb's free energy decreases with increase in temperature.

From Table 2, we see that at the given temperature, the value of $r_i^*, \Delta G_i^*$ and $\ln J_i^*$ are found to increase with increase in applied external electric field. But, at a fixed external electric field, the values of $r_i^*, \Delta G_i^*$ and $\ln J_i^*$ decrease with increase in temperature.

For example. at electric field 10 esu, the radii are $29.50\mathrm{x}10^{-9}$, $29.38\mathrm{x}10^{-9}$ and $29.25\mathrm{x}10^{-9}$ cm corresponding to the temperatures 263, 273 and 283 K, respectively. Similarly, at the same electric field, the critical Gibb's free energy for formation of ice nucleus are $36.48\mathrm{x}10^{-14}$, $36.16\mathrm{x}10^{-14}$, and $35.85\mathrm{x}10^{-14}$ erg at temperatures 263, 273 and 283 K, respectively. Similar variations are evident for $ln\ J_i^*$.

Table 3 represents the calculated values of $r_i^*, \Delta G_i^*$ and $\ln J_i^*$ as the function of relaxation time at electric field 10 esu for temperature 263, 273 and 283 K. These values have been found to increase with increase in relaxation time for given temperature. But at a given relaxation time, these value decrease with increase in temperature.

From above discussion we may conclude that the nucleation parameters; critical radius. Gibb's free energy are highly decreased, while nucleation rate is increased with increase in temperature. Thus, the nuclei of smaller size requiring less Gibb's free energy are produced in large numbers i.e. nucleation rate is enhanced.

CONCLUSION

From above study, it may be concluded that the nucleation process is accelerated in presence of external electric field as compared with the homogeneous nucleation process. In clouds and thunderstorms, electric fields are generated due to lightning discharges.

Acknowledgement : With thank Dr.S.K. Mishra ,Principal, D.B.S. College, Kanpur for his helpful suggestions that improved this study .Thank are also acknowledged to Director N.P.L. New Delhi for providing the library facilities.

REFERENCES

1. M.A. Abbas, and J. Latham : The electrofreezing of supercooled water drops, J. Meteor. Soc. Japan, 47 (1969) 67-74.
2. F.F. Abraham : A re-examination of homogeneous nucleation theory: Thermodynamic aspects, J. Atmos. Sci., 25 (1968) 47-53.
3. J.T. Bartlett, A.P. van den Heuvel and B.J. Mason : The growth of ice crystals in an electric field, Z. Angew. Math. Phys., 14 (1963) 599-610.
4. F.C. Collins : Time lag in spontaneous nucleation due to non-steady state effects, Zeitschrift für Elektrochemie, 59 (1955) 404-407.
5. C.G. Deshpande and A.K. Kamra : Diurnal variations of the atmospheric electric field and conductivity at Maitri, Antarctica, 1. Geophys. Res., 106 (D13) (2001) 14207-14218.
6. J.B. Doolittle and G. Vali : Heterogeneous freezing nucleation in electric fields, J. Atmos. Sci., 32 (1975) 375-379.
7. L. Dufour : Uber das Gefrieren das Wassers und ubber die Bildung des Hagels, Poggendorfs, Ann. Physik, 114 (1861) 530-554.

8. L.F. Evans : The growth and fragmentation of the crystals in an electric field, J. Atmos. Sci., 30 (1973) 1657-1664.

9. T.G. Gabarashvili and N.V. Gliki : Origination of the ice phase in supercooled water under the influence of electrically charged crystals, Izv. Atmos. Oceanic Phys., 3 (1967) 570-574.

10. A.K. Kamra, C.G. Deshpande and Y. Gopalkrishnan : Challenge to the assumption of the unitary diurnal variation of the atmospheric electric field based on observations in the Indian Ocean, Bay of Bengal and Arbian Sea. J. Geophys. Res., 99 (1994) 21043 - 21050.

11. A. Kantrowitz : Nucleation in very rapid vapour expansions, J. Chem. Phys., 19 (1951) 1097-1100.

12. B. Karcher and U. Lohmann : A parameterization of cirrus cloud formation: Homogeneous freezing of supercooled aerosols, J. Geophys. Res., 107 (D2) (2002) AAC.4-1 - AAC.4-10.

13. L.B. Loeb : A tentative explanation of the electric field effect on the freezing of supercooled water drops, J. Geophys. Res., 68 (1963) 4475-4476.

14. T.C. Marshall and M. Stolzenburg : Electrical energy constraints on lightning, J. Geophys. Res., 107(D7) (2002) ACLl-1 - ACL1-13.

15. B.J. Mason : The Physics of Cloud, Oxford University Press, London (1972).

16. J. Maybank and N.N. Barthakur : Growth and destruction of ice filaments in an electric field, Nature, 216 (1967) 50-52.

17. M. Mikki, V.A. Rakov, K.J. Rambo, G.H. Schnetzer and M.A. Uman : Electric fields near triggered lightning channels measured with Pockels sensors, J. Geophys. Res., 107 (D16) (2002) ACL.2-1 - ACL.2-11.

18. G. Murino : Influence of electric fields on condensation of water vapour, S. Afr. Tydskr. Fis., 2 (1979) 113-115.

19. R.F. Probstein : Time lag in the self nucleation of a supersaturated vapour, J. Chem. Phys., 19 (1951) 619-626.

20. H.R. Pruppacher : The effect of an external electric field on the supercooling of water drops, J. Geophys. Res., 68 (1963) 4463-4474.

21. H.R. Pruppacher : Electro-freezing of supercooled water, Pure Appl. Geophys., 104 (1973) 623-633.

22. W. Rau : Eiskeimbildung durch dielektrische polarization, Z.F. Naturforsch, 6a (1951) 649-657.

23. M. Roulleau, L.T. Evans, and N. Fukuta : The electrical nucleation of ice in supercooled clouds, J. Atmos. Sci., 28 (1971) 737-740.

24. W.D. Rust and R.J. Trapp : Initial balloon soundings of the electrical field in winter nimbostratus clouds in the U.S.A., Geophys. Res. Lett., 29 (2002) 20-1-20-4.

25. R.W. Salt : Effect of electrostatic field on freezing of supercooled water and insects, Science, 133 (1961) 458.

26. A.R. Sharma, N. Singh and S.D. Pandey : On the equivalence of external electric field to the supersaturation ratio in water vapour condensation, Ind. J. Radio & Space Phys., 21 (1992) 218-228.

27. N. Singh, J. Rai and N.C. Varshneya : The effect of external electric field on relaxation time in nucleation process of water vapour condensation and ice glaciation, Ann. Geophys., 42(1) (1986) 37-44.

28. M.H. Smith, R.F. Griffiths and J. Latham : The freezing of raindrops falling through strong electric fields, Quart. J. Roy. Meteorol. Soc., 97 (1971) 495-505.

29. H. Wakeshima : Time lag in the self nucleation, J. Chem. Phys., 22. (1954) 1614-1615.

30. P.K. Wang : The influence of atmospheric electricity on the precipitation, Seventh International Conference on Atmospheric Electricity, June 3-8, 1984. Albany, N.Y., U.S.A., 1984.

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 89-96

KĀTYĀYNA ŚULB-SŪTRA

VIRENDRA ARORA* AND NIDHI HANDA**

(Received 06.10.2004 and in revised form 06.10.2004)

ABSTRACT

Mature Science offers a new tool of interpretation to the histrorians who is then able to discover new results in those original works that they have already studied. Here Kandika 3 of Kātyāyana Śulba Sūtra deals with some gemetrical figures and their transformations.

Key Words : KSS- Kātyāyana Śulba Sūtra, KSS-3/1 - first Verse of their Kandika of KSS.

INTRODUCTION

This paper aims at the conversion of area i.e. the conversion from rectangle into square and vice-versa ; circle in to square and vice-versa of various dimensions. It focuses how the four sided complicated structures have been simplified in the Śulba Sūtra. Here the endeavor has also been made to highlight the method of construction and proof of each sutra. This paper is discussed under the following topics.

- Extraction of a small square from large one.
- Conversion of a Rectangle into square.
- Transformation of a Lengthy rectengle in to a square.
- Transformation of a square in to rectangle.
- Some triplets for areas.
- Transformation of square in to circle and circle in to suquare.

EXTRACTION OF A SMALL SQURE FORM LARGE ONE

चतुरस्राच्चतुरस्रं निर्जिहीर्षन् यावन्निर्जिहीर्षे त्तावदुभयतोऽपच्छिद्य शङ्कू
निखाय पार्श्वमानी कृत्वा पार्श्वमानी सम्मितामक्ष्णयां तत्रोपसंहरति
समासेऽपच्छेद: सा करण्येष निह्रसि: ॥ 3/1 ॥

This means mark the length of the side of a small square on two opposite sides of a big one. Join the two points by a straight line Fix one point of straight line as center and other as moving point Extend the moving point to the base of big square. Now the large portion of base will be side of a required square which will be remaining area after subtracting the small square form the large one. Thus, small square, EFGH has been extracted form the given large square ABCD

---

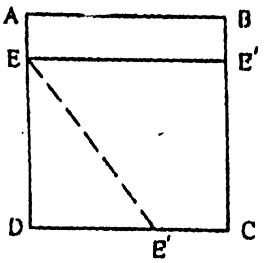
* Dept. of Mathematics and Statistics, Gurukul Kangri University, Hardwar.
** Dept. of Mathematics and Statistics, Kanya Gurukul Mahavidyalaya, Hardwar

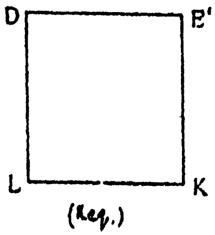
In right-angled triangle EDE'

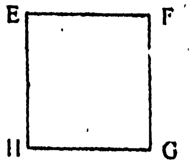
$ED^2 + DE'^2 = EE'^2$

$ED^2 + EF^2 = AB^2$ $(EE' = AB, DE' = EF)$

$ED^2 = AB^2 - EF^2$







CONVERSION OF A RECTANGLE INTO A SQUARE

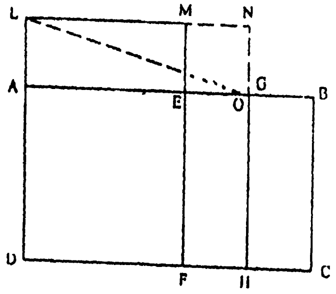
दीर्घचतुरस्रं समचतुरस्रं चिकीर्षन्मध्ये तिर्यगपच्छिद्यान्यतरद्विभज्ये
तरत्पुरं-स्ताद्दक्षिणतश्चोपदध्याच्छेषमागन्तुना पूरयेत्तस्योक्तो निर्ह्रास ॥ 3/2 ॥

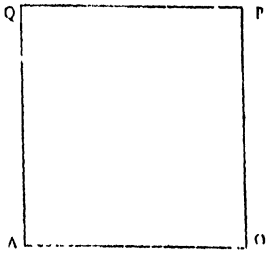
To Transfer a rectangle ABCD into square draw in line EF bisecting the lines AB and DC. Now in Sq. EBCF draw a line GH bisecting sides EB and FC. Place rect. GBCH at AE and we have LMEA. Now Complete a big square LNHD by adding sq. MNGE.

Mathematically

$AO^2 = OL^2 - AL^2$ $(OL = LN)$

$AO^2 = LN^2 - AL^2$





Thus AO^2 represent the square AOPQ. Here square MNGE is being subtracted form the square LNHD to get the required square AOPQ.

TRANSFORMATION OF A LENGTHY RECTENGLE IN TO A SQUARE

अतिदीर्घञ्चे चिर्यङ् मान्याऽपच्छिद्यैक समासेन समस्य शेषं यथा योगमुपसहरे दिदत्येक समासः ॥ ३/३॥

A long rectangle ABCD is divided in to equal squares ABOL, LOPM, MPQN and one small rectangle DCQN.

In fig. 2 a square will represent area of three squares (which is made on the hypotenuse of right angled triangle.)

Cut an arc GT = FC in GR taking G as center. Join OT.

In fig. 3 a square made on OT will represent the req. area of given long rectangle.

i.e. $AB^2 + LO^2 + MP^2 + NF^2 + FQ^2 = OT^2$

Since $AB^2 + LO^2 + MP^2 = OG^2$

Again in Δ OGT

$OG^2 + GT^2 = OT^2$

$(GT = CF)$

$(CF^2 = FQ^2 + NF^2)$

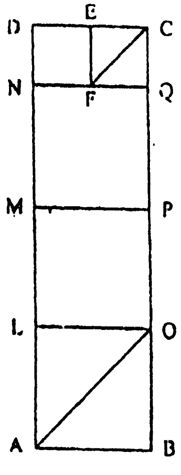


FIG. 1

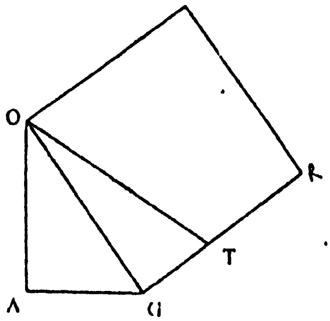


FIG. 2

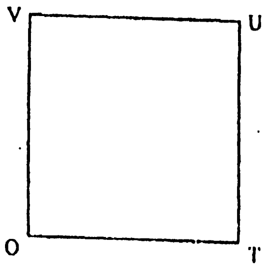


FIG. 3

Hence area of required square OTUV = area of rectangle ABCD

TRANSFORATION OF A SQUARE IN TO RECTANGLE

समचतुरस्रं दीर्घचतुरस्रं चिकीर्षन्मध्येऽक्ष्णयाऽपच्छिद्य विभज्येतरत्
पुरस्तादुत्तरतश्चोपदध्याद, विषमं चेद् यथा-योगमुपसंहरेदिति व्यस: ॥ 3/4 ॥

Divide a square by diagonal into two equal parts. Again divide one of the parts in to two so that we have two isosceles Δs

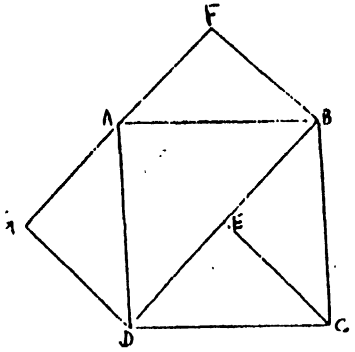
Place these two triangles on the opposite corners of the adjacent sides.

Δ DEC ≡ Δ BEC

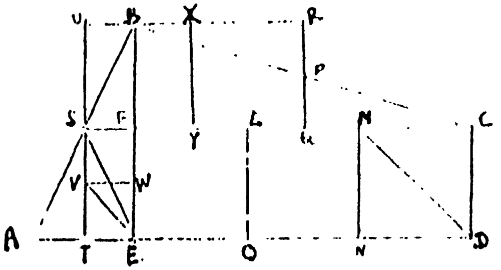
Place Δ BEC on Δ FAB.

Similarly put Δ EDC on Δ GDA.

GDBF is a required rectangle.



Some examples have been shown to describe this situation by Ācaryā Kātyāyana. According to Ācaryā Kātyāyana four sidea figure (quadrilateral) can be converted in to rectangle by substituting and eliminating some area. The figure may be divided in to small square then converted in to rectangle. Where the required figure (which is rectangle) do not come in to existence there only it is divided in to squares, for example in figure 1.



SOME TRIPLETS FOR AREA

प्रमाणं चतुरसमादेशादन्यत् ॥ 3/5 ॥

Unless or untill there is specific direction of unit, a square unit is to be accepted. If there is some specific direction then only act accordingly.

द्वि प्रमाणा चतु:करणी, त्रिप्रामाणा
नवकरणी, चतुः प्रमाणा षोडशकरणी ॥ 3/6॥

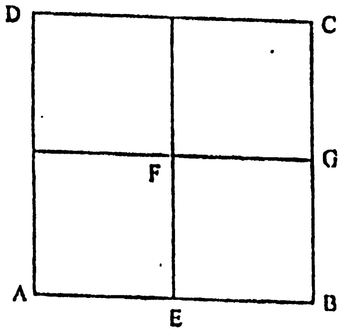
As much the length so much squared area will be got, because same digits will be multipiled with each other. If measure of unit is two then its area will be four unit of square, If three then nine, if four then 16 and so on. Thus this is a little told about area.

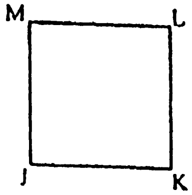
यावत् प्रमाणा रज्जुर्भुवति तावन्तस्तावन्तो
वर्गा भवन्ति तान्समस्येत् ॥ 3/7 ॥

As many units of measure in a chord so many squares of those units will be. Add all those squares then area will be got. If two units of measure taken, four squares will be formed for area (2×2 = 4). If five units are there then 25 squares wil be formed (5×5 = 25). Similarly other can be produced.

अर्धप्रमाणेन पादप्रमाणं, विधीयते ॥ 3/8 ॥

But when half measure is taken then what will square of that space be ? If half the measure is squared it is fourth part of complete spare $(1/2)^2 = 1/4$. If the measure is 24 units. Its half is 12 Square of 24 is 576 and that of 12 is 144. Thus 144 is one fourth of 576. This way for other units of measure can be applied.



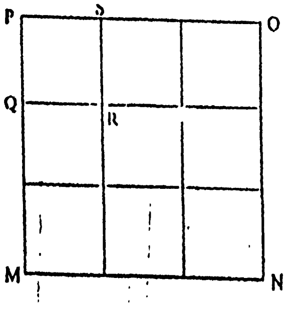


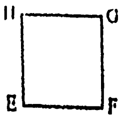
Example

Let us take ABCD whose measures is 24 × 24 sq. units. JKLM is a small square whose measure is 144 sq. units and this is fourth part of the square made by ABCD which is also equal to EBGF.

तृतीयेन नवर्मोऽशः ॥ 3/9 ॥

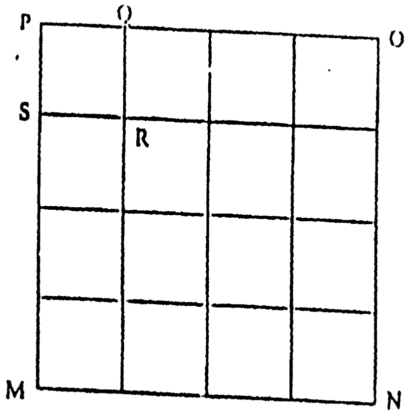
The square formed by one third the length of given measure is ninth part of total area of the given measure of unit. e.g. MP is given : 3 unit of measure, its third part is taken and square EFGH is completed. Now MNOP square is being completed and we see that PQRS is ninth part of MNOP. Area of this part PQRS is same as that of EFGH.

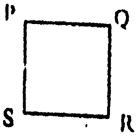




चतुर्थेन षोडशी कला ॥ 3/10 ॥

Area of the sqare which is one fourth of the length of a measure is sixteenth part of the square formed by full length of a measure. How can this area be separately deducted (nirhavasa).
[Consult Kss 3/1].





एष निर्ह्रासस्तस्य पुरस्तादुक्तं शास्त्रम् ॥ 3/11 ॥

To deduct fourth part from 2 units of measure, ninth part from 3 units of measure, sixteenth part from 4 units of measure of side and so on, is already explained.

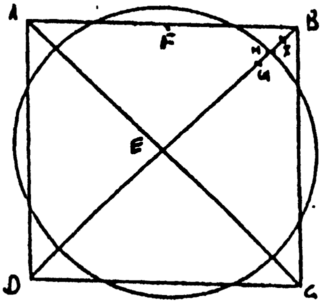
यावत्प्रमाणा रज्जुर्भवतीति विवृद्धे ह्रासाभवति ॥ 3/12 ॥

Deduction and entargement are made according to unit length of the chord. If lenght of chord is nine unit then its third part can be deducted. Similarly fourth part from sixteen units.

TRANSFORMATION OF SQUARE INTO CIRCLE & VICE-VERSA

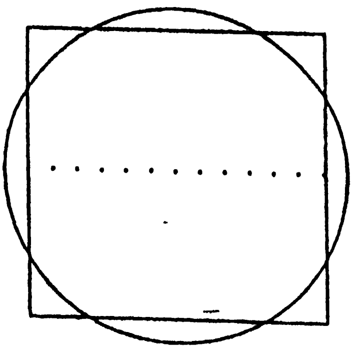
चतुरस्रं मण्डलं चिकीर्षन्मध्यादसे निपात्य पार्श्वत: परिलिख्य तत्र
यदतिरिक्तं भवति तस्य-तृतीयेन सह मण्डलं परिलिखेत्स समाधि: ।। 3/13 ।।

Transform ABCD square into a circle. Draw diagonal AC,BD from angle A and angle D. E is their meetting point, cut half of AB at F cut off on arc EG = FB from E. Divide GB into three parts GH. HI, IB. Now leave two parts and with centre EH draw a circle whose area will be equal to that square (for details see 2,p.65).



मण्डलं चतुरस्रं चिकीर्षन् विष्कम्भं पंचदशभागान्
कृत्वा द्ववुद्धरेच्छेष: करणी ।। 3/14।।

To convert a circle into square divide diameter into 15 parts.



Give up two parts out of them. With the help of remaining chord which is of 13 parts, draw a square. Now area of a square will be equal to that of the given circle (fordetails see 2, P 65).

After discussing the varous s̄utras composed by Kātyāyana regarding the conversion of area it is concluded that in vedic age, there were various figures of different dimensions. It is also mentioned that in those days the device of dealing with the mathematical problems were very much in practice.

REFERENCES

1. Kātyāyana Śulba, edited with Karka's Bhasya and Mahidhara's vriti in the Kasi Sanskrit series, Banaras, 1936
2. Khadikar, S.D. (ed.), Kātyāyana Śulba Sūtra, Vaidika Saṃsodhana Mandala,Puna, 1974
3. R.C.D. Sharma, महार्षिकात्यायनप्रणीतम्, कात्यायन शुल्बसूत्रम, नाग प्रकाशक, दिल्ली (1993)
4. S. N. Sen & A.K. Bag, The Śulba Sūtra of Baūdhayana, Āpastamba, Kātyāyan and Manava, with text, English Translation and commentary, Indian National Science Academy, New Delhi, 1983.

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 97-104

New Coincidences and Fixed Points of Reciprocally Continuous and Compatible Hybrid Maps

S. L. SINGH* and AMAL M. HASHIM**

(Received 20.09.2004 and after revision 12.10.2004)

ABSTRACT

It is proved that a pair of reciprocally continuous and non-vacuously compatible single-valued and multivalued maps on metric spaces possesses a coincidence. This result is applied to obtain new general coincidence and fixed point theorems for single-valued and multivalued maps in metric spaces under tight minimal conditions

Mathematics subject classifications (2000): 54H25, 47H10

Keywords and phrases: Coincidence point, fixed point, hybrid maps, reciprocally continuous, compatible maps.

INTRODUCTION

The notion of compatible maps, introduced by G. Jungck [7] (also called asymptotically commuting maps, cf. [24]) has proved useful for generalizing known results in the context of metric fixed point theory for continuous single-valued and multivalued maps (see, for instance, [1] - [2], [4], [7], [8], [11], [16], and [22]).

Recently, reciprocal continuity for a pair of (discontinuous) single-valued maps (*resp.* hybrid pair of maps) has been introduced in [16] (*resp.* [21]).

In this paper we obtain coincidence and fixed point theorems for a hybrid quadruple of maps on a metric space satisfying very general contractive type conditions which include several general conditions studied by Beg and Azam [1], Ćirić [3], Jungck [8], Kaneko [10], [11], Kaneko and Sessa [12], Lohani and Badshah [13], Rhoades *et al.* [17], Tan and Minh [23] and others.

---

*Dept. of Maths & Stat. Gurukula Kangri University, Hardwar 249404
Mailing address: 21, Govind Nagar, Reshikesh 249201, India. **E- mail**: vedicmri@sancharnet.in.
****Permanent address**: Dept. of Mathematics, College of Science, Univ. of Basrah, Iraq.
E-mail: amalmhashim@yahoo.com.

PRELIMINARIES

Consistent with [1], [14], [15] and [20], we will use the following notations, where (X, d) is a metric space and $CL(X)$ is a collection of all nonempty closed subsets of X. Further, $d(A, B)$ denotes the ordinary distance between nonempty subsets of A and B of X, while $d(x, B)$ stands for $d(A, B)$ when $A = \{x\}$. The distance function H is called the generalized Hausdorff metric for $CL(X)$ induced by the metric d of X.

Definition 2.1 (Singh and Mishra [21]). The maps $T: X \to CL(X)$ and $f: X \to X$ are reciprocally continuous (r. c.) on X (*resp.* at $t \in X$) if and only if $fTx \in CL(X)$ for each $x \in X$ (*resp.* $fTt \in CL(X)$) and $\lim\limits_{n\to\infty} fTx_n = fM$, $\lim\limits_{n\to\infty} Tfx_n = Tt$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n = M \in CL(X),\ \lim_{n\to\infty} fx_n = t \in M.$$

For self-maps $f, g: X \to X$, this definition due to Pant [16] reads: f and g are r.c if and only if $\lim\limits_{n\to\infty} gfx_n = gt$ and $\lim\limits_{n\to\infty} fgx_n = ft$ whenever $\{x_n\} \subset X$ is such that $\lim\limits_{n\to\infty} gx_n = \lim\limits_{n\to\infty} fx_n = t \in X$. Clearly any continuous pair is reciprocally continuous but the converse is not true (Singh and Mishra [21, Example 2.3]). We remark that reciprocally continuous maps may be discontinuous even at their common fixed points (see [21]). For continuity of multivalued maps at their fixed and common fixed points, refer to [5].

The following definition is due to Kaneko and Sessa [12] and Beg and Azam [1] when $T: X \to CB(X)$.

Definition 2.2 (Singh and Mishra [21]). The maps $T: X \to CL(X)$ and $f: X \to X$ are compatible if and only if $fTx \in CL(X)$ for each $x \in X$ and $\lim_{n\to\infty} H(Tfx_n, fTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n, = M \in CL(X) \text{ and } \lim_{n\to\infty} fx_n = t \in M.$$

Evidently commuting maps T, f (i.e., when $fTx = Tfx$, $x \in X$) are weakly commuting (i.e., $H(Tfx, fTx) \leq d(fx, Tx)$, $x \in X$, [10] and [19]), weakly commuting maps are compatible, and compatible T, f are weakly compatible (i.e., when $fTx = Tfx$, whenever $fx \in Tx$, see Jungck and Rhoades [9]), but the reverse implication is not true. For excellent discussion on the role of weak compatibility in fixed point considerations, refer to Jungck and Rhoades [*op. cit.*].

Definition 2.3 (Itoh and Takahashi [10]). The maps $T: X \to CL(X)$ and $f: X \to X$ are (IT)-commuting at a point $v \in X$ if $fTv \subset Tfv$ (cf. [21, page 629]).

Remark 2.4.

(i) (IT)-commutativity of a hybrid pair (T, f) at a point v is more general than its compatibility (cf. [21, Example 1]) and weak compatibility at the point v.

(ii) Continuous and commuting pair of hybrid maps (T, f) of a complete metric space need not have a coincidence; for example $Tx = \{1 + x\}$ and $fx = x$, $x \in [0, \infty)$, (see [21]).

(iii) The nonvacuous compatibility of T and f implies the existence of at least a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Tx_n = M \in CL(X)$ and $\lim_{n\to\infty} fx_n = t \in X$). For details see [21].

MAIN RESULTS

The following is the main result. In all that follows, $C(S, A)$ stands for the collection of coincidence points of S and A, that is, $C(S, A) = \{v;\ Sv \in Av\}$.

Theorem 3.1. Let (X, d) be a metric space and $A_i: X \to CL(X)$, i = 1, 2, ... and $S, T: X \to X$ such that

(i) $A_1X \subset TX$ and the pair (A, S) is reciprocally continuous and nonvacuously compatible,

(ii) $H(A_1x, A_iy) < (m_{1i}(x, y))$, $i > 1$ and $x, y \in X$ when $m_{1i}(x, y) > 0$, where $(m_{1i}(x, y)) = \max\{d(Sx, A_1x), d(Ty, A_iy), d(Sx, A_iy), d(Ty, A_1x), d(Sx, Ty)\}$.

Then $C(S, A_i)$ and (T, A_i) are nonempty. Further,

(I) S and A_1 have a common fixed point Su, provided $SSu = Su$ for some $u \in C(S, A_1)$;

(II) T and A_i have a common fixed point Tw, provided $TTw = Tw$ and T, A_i are (IT)-commuting at $w \in C(T, A_i)$;

(III) S, T, A and A_i have a common fixed point, provided (i) and (ii) both are true.

Proof. Since A_1 and S are nonvacuously compatible, there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} A_1x_n = M \in CL(X)$ and $\lim_{n\to\infty} Sx_n = u \in M$ and $\lim_{n\to\infty} H(A_1Sx_n, SA_1x_n) = 0$. This, in view of r. c. of A_1 and S, yields $H(A_1u, SM) = 0$, and $A_1u = SM$. Now $u \in M$ implies $Su \in A_1u$. Therefore $C(S, A_1)$ is nonempty. Since $A_1X \subset TX$, there exists a point $w \in X$ such that $Su = Tw \in A_1u$.

If $A_iw \neq Tw$, then

$$\begin{aligned} d(Tw, A_i w) \le H(A_1 u, A_i w) &< \max\{d(Su, A_1 u), d(Tw, A_i w), d(Tw, A_1 u), \\ &\qquad d(Su, A_i w), d(Tw, Su)\} \\ &= d(Tw, A_i w), \end{aligned}$$

a contradiction. This implies that $Tw \in A_i w$, and $C(T, A_i)$ is nonempty.
Further, $Su = SSu$ and the (IT)-commutativity of A and S at $u \in C(S, A)$ imply that $Su \in SA_1 u = A_1 Su$. So Su is a common fixed point of A and S.
The proof of (II) is similar, and (III) is immediate.

Corollary 3.2. Let (X, d) be a metric space, $A, B: X \to CL(X)$ and $S, T: X \to X$ such that

(iii) $AX \subset TX$ and the pair (S, A) is reciprocally continuous and nonvacuously compatible;

(iv) $H(Ax, By) \le q(M(x, y))$ for each x, y in X and $q \in (0, 1)$, where
$M(x, y) = \max\{d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Sx), d(Sx, Ty)\}$.
Then $C(S, A)$ and $C(T, B)$ are nonempty. Further,

(Ia) S and A have a common fixed point Su, provided $SSu = Su$ for some $u \in (S, A)$;

(IIa) T and B have a common fixed point Tw, provided $TTw = Tw$ and T, B are (IT)-commuting at $w \in C(T, B)$;

(IIIa) S, T, A and B have a common fixed point, provided (Ia) and (IIa) both are true.

Proof. It may be completed following the proof of Theorem 3.1.

We remark that Theorem 3.1 with all maps single-valued presents a generalization of the main result of Pant [16, page 148]. Further, the above results extend some of the results of Singh and Kumar [18] to a quadruplet of hybrid maps.

Theorem 3.3. Corollary 3.2 with (iv) replaced by following :

(v) H(Ax, By) < $L(x, y)$ for all x, y in X when $L(x, y) > 0$, where

$$L(x, y) = \frac{\alpha\, d(Ty, By)[1 + d(Sx, Ax)]}{1 + d(Sx, Ty)} + \beta\, [\, d(Ty, By) + (Sx, Ax)]$$
$$+\ \gamma[d(Sx, By) + d(Ty, Ax)] + \delta\, d(Sx, Ty),$$

where α, β, γ, $\delta \geq 0$ and $\alpha + \beta + 2\gamma + \delta < 1$.

Proof. It may be completed following the proof of Theorem 3.1.

Remark 3.4. Theorem 3.3 generalizes a result of Lohani and Badshah [13] and others.

REFERENCES

1. Ismat Beg and Akbar Azam: Fixed points of asymptotically regular multivalued mappings, J. Austral. Math. Soc. Ser..A 53 (1992), 313-326. **MR1187851 (93m:54033).**

2. T. H. Chang: Common fixed point Theorems for multivalued mapping, Math. Japon, 41(1995), No. 2, 311-320. **MR1326964 (96a:54057)**.

3. Lj. B. Ćirić: A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267-273. **MR0356011 (50 #8484).**

4. O. Hădžic: Common fixed point theorems for single and multivalued mappings, Univ. Novom Sadu Zb. Rad. Prirod Mat. Fak. Ser Mat., 18(1988), No. 2, 145-151._**MR1100465 (92d:54060).**

5. T. H. Hicks and B. E. Rhoades: Fixed points and continuity for multivalued mappings, Internat. J. Math. Math. Sci. 15(1992), 15-30. **MR1143924 (93b:54043).**

6. S. Itoh and W. Takahashi: Single-valued mappings, multivalued mappings and fixed point theorems, J. Math. Anal. Appl. 59 (1977), 514-521. **MR0454752 (56#13000).**

7. G. Jungck: Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986), 771-779. **MR0939083 (89h:54029).**

8. G. Jungck: Common fixed points for commuting and compatible maps on compacta, Proc. Amer. Math. Soc. 103 (1988), 977-983. **MR0947693 (89h:54030).**

9. G. Jungck and B. E. Rhoades: Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29 (1998), 227-238. **MR1617919.**

10. H. Kaneko: A report on general contractive type conditions for multivalued mappings, Math. Japon. 33 (1988), 543-550. **MR0961199 (90a:47144).**

11. H. Kaneko: A common fixed point of weakly commuting multi-valued mappings. Math. Japon. 33 (1988), no. 5, 741–744. **MR0972386 (90h:54055).**

12. H. Kaneko and S. Sessa, Fixed point theorems for compatible multivalued and single-valued mappings, Internat. J. Math. Math. Sci. 12 (1989), 257-262. **MR0994907 (90i:54097)**

13. P. C. Lohani and V. H. Badshah: Compatible mappings and common fixed point for four mappings, J. Natur. Phy. Sci. 11 (1997), 1-12. **MR1659342.**

14. S. B. Nadler, Jr.: Multivalued contraction mappings, Pacific J. Math. 30 (1969), 475-488. **MR0254828 (40 #8035).**

15. S. A. Naimpally, S. L. Singh and J. H. M. Whitfield: Coincidence theorems for hybrid contractions, Math. Nachr. 127 (1986), 177-180. **MR0861724 (87m:54118).**

16. R. P. Pant: A common fixed point theorem under a new condition, Indian J. Pure Appl. Math. 30 (1999), 147-152. **MR1681592.**

17. B. E. Rhoades S. L. Singh and Chitra Kulshrestha: Coincidence theorems for some multivalued mappings, Internat. J. Math. Math. Sci. 7 (1984), 429-434. **MR0771590 (86d:54073).**

18. S. L. Singh, Ashish Kumar and Amal M. Hashim: Fixed point of contractive maps (2004), (reprint).

19. S. L. Singh, K. S. Ha and Y. J. Cho: Coincidence and fixed points of nonlinear hybrid contractions, Internat. J. Math. Math. Sci. 12 (1989), 247-256. **MR0994906 (90g:54042).**

20. S. L. Singh and S. N. Mishra:Coincidences and fixed points of nonself hybrid contractions, J. Math. Anal. Appl. 256 (2001), 486-497. **MR1821752.**

21. S. L. Singh and S. N. Mishra: Coincidences and fixed points of reciprocally continuous and compatible hybrid maps, Internat. J. Math. Math Sci. 30 (2002), 627-635. **MR1918077 (2003f:54102).**

22. S. L. Singh and S. N. Mishra: On general hybrid contractions, J. Austral. Math. Soc. Ser. A 66 (1999), 244-254. **MR1671885 (2000b:54057).**

23. D. H. Tan and N. A. Minh: Some fixed point theorems for mappings of contractive type, Acta Math. Vietnam. 3(1978), 24-42. **MR0527475(80d:54054).**

24. B. M. L. Tivari and S. L. Singh: A note on recent generalizations of Jungck contraction principle, J. UPGC. Acad. Soc. 3 (1986), 13-18. **MR0914072 (88k:54081).**

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004) 105-116

COINCIDENCE AND FIXED POINT THEOREMS FOR HYBRID CONTRACTIONS

S. L. Singh* and Ritu Arora**

(Received 5.10.2004 and after revision 19.10.2004)

ABSTRACT

The purpose of this paper is to obtain coincidence and fixed point theorems for a pair of hybrid contracting maps.

Mathematics Subject Classifications (2000): 54H25, 54C60, 47H10.
Key words and phrases: Coincidence point, fixed point, hybrid contracting maps, hybrid contraction.

1. INTRODUCTION

Consistent with [1], [4], [11], [16], [18], [20] and [23], we will use the following notations, where (X, d) is a metric space. Let $CL(X)$ denote the collection of all nonempty closed subsets of X. The distance function H on $CL(X)$ is called the generalized Hausdorff metric induced by the metric d of X. Further, let $d(A, B)$ denote the ordinary distance between nonempty subsets A and B of X, while $d(A, x)$ stands for $d(A, B)$ when B is the singleton $\{x\}$.

Let $T : X \to CL(X)$ be a multivalued map and $f : X \to X$ a single

---

* 21, Govind Nagar, Rishikesh 249201 India; Email: sh42@rediffmail.com
** Department of Holistic Medicine, Himalayan Institute Hospital Trust, Jolly Grant, Dehradun 248140

-valued map. Consider the following condition essentially due to Singh and Kulshrestha [21].

$$H(Tx, Ty) \leq q.\max\{d(fx, fy), d(fx, Tx), (fy, Ty), [d(fx, Ty) + d(fy, Tx)]/2\} \qquad \text{(SK)}$$

for all $x, y \in X$, where $q \in (0, 1)$.

We remark that (SK) with $f =$ the identity map on X is Ćirić's generalized multivalued contraction (Ćirić [4]), which in turn includes the well-known Nadler multivalued contraction (Nadler, Jr. [16], see also [1], [4], [9]-[13], [15], [23] and [25]).

The study of hybrid contracting maps involving single-valued and multivalued maps on metric spaces was initiated independently by Bhaskaran and Subrahmanyam [2], Hadžić [5], Kaneko [9], Kubiak [15], Naimpally [17] *et al.* and Singh and Kulshrestha [21]. (Here, according to Singh and Mishra [22], "hybrid contracting maps" means a pair of hybrid contraction or nonexpansive or contractive). Indeed, Singh and Kulshrestha [*op. cit.*] (see also [12], [19] and [22]) showed that T and f satisfying $T(X) \subseteq f(X)$ and (SK) have a coincidence, that is, there exists a point $z \in X$ such that $fz \in Tz$ when $f(X)$ is a complete subspace of X. This result is obviously true when, instead of $f(X)$, $T(X)$ is a complete subspace of X . For an immediate excellent generalization of this result one may refer to Rhoades *et al.* [18].

In all that follows, Y is an arbitrary nonempty set and (X, d) a metric space. Following Liu *et al.* [14], Singh and Mishra [23] and Tan *et al.* [26], we shall consider the following condition for $f: Y \to X$ and $T: Y \to CL(X)$.

$$H^2(Tx, Ty) \leq q.m(x, y), \tag{1.1}$$

where $q \in (0, 1)$ and

$$\begin{aligned} m(x, y) := \max \{ & d^2(fx, fy), d(fx, fy).d(fx, Tx), d(fx, fy).d(fy, Ty), \\ & d(fx, fy).[d(fx, Ty) + d(fy, Tx)]/2, d(fx, Tx).d(fy, Ty), \\ & d(fx, Tx).[d(fx, Ty) + d(fy, Tx)]/2, d(fy, Ty).[d(fx, Ty) \\ & + d(fy, Tx)]/2, d(fx, Ty).d(fy, Tx)\}. \end{aligned}$$

Our main result is under the condition (1.1) (cf. Theorem 3). Following Rhoades *et al.* [18], we present generalized versions of this result as well.

2. RESULTS

DEFINITION 1. (Itoh and Takahashi [6], see also Singh and Mishra [24, p. 488]). Let Y be a nonempty set, $f: Y \to Y$ and $T: Y \to 2^Y$, the collection of all nonempty subsets of Y. Then the hybrid pair (T, f) is IT-commuting at $x \in Y$ if $fTx \subseteq Tfx$ for each $x \in Y$. (This formulation in [25] is correct with the interchange of symbols for single-valued and mutivalued maps).

We shall need the following result, which is a minor variant of a lemma due to Ćirić [4].

LEMMA 2. Let $A, B \in CL(X)$. Then for an $x \in A$ and for some q and k in (0, 1), there exists a $y \in B$ such that

$$d^2(x, y) \leq q^{-k} H^2(A, B).$$

THEOREM 3. Let Y be an arbitrary nonempty set and (X, d) a metric space. Let $T : Y \rightarrow CL(X)$ and $f : Y \rightarrow X$ be such that $T(Y) \subseteq f(Y)$ and (1.1) holds for all $x, y \in Y$. If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence. Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that

(I) $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$;

(II) $\{fx_n\}$ converges to fz for some $z \in Y$, and $fz \in Tz$,
that is, T and f have a coincidence at z; and

(III) $d(fx_n, fz) \leq [\beta^n/(1-\beta)].d(fx_0, fx_1)$, where $\beta = q^{1-k}[1 + \sqrt{(1 + 8q^{-1+k})}]/4$
for some $k \in (0, 1)$.

Further, if $Y = X$ and the pair (T, f) is IT-commuting at z then T and f have a common fixed point provided that $ffz = fz$.

PROOF. Pick a point x_0 in Y. Let k be a positive number such that $k < 1$. Following Singh and Kulshrestha [21] (see also [12]), we construct sequences $\{x_n\} \subseteq Y$ and $\{fx_n\} \subseteq X$ in the following manner.

Since $T(Y) \subseteq f(Y)$, we may choose a point $x_1 \in Y$ such that $fx_1 \in Tx_0$. If $Tx_0 = Tx_1$ then $x_1 = z$ is a coincidence point of T and f, and we are done. So assume that $Tx_0 \neq Tx_1$ and choose $x_2 \in Y$ such that $fx_2 \in Tx_1$ and, by Lemma 2,

$$d^2(fx_1, fx_2) \leq q^{-k} H^2(Tx_0, Tx_1).$$

If $Tx_1 = Tx_2$, then x_2 becomes a coincidence point of T and f. If not, continue the process. In general, if $Tx_n \neq Tx_{n+1}$, we choose $fx_{n+2} \in Tx_{n+1}$ such that

$$d^2(fx_{n+1}, fx_{n+2}) \leq q^{-k} H^2(Tx_n, Tx_{n+1}).$$

Then by (1.1),

$$\begin{aligned} d^2(fx_{n+1}, fx_{n+2}) &\leq q^{-k}H^2(Tx_n, Tx_{n+1}) \\ &\leq q^{1-k}\max \{d^2(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}).d(fx_n, Tx_n), d(fx_n, fx_{n+1}).d(fx_{n+1}, Tx_{n+1}), \\ &\quad d(fx_n, fx_{n+1}).[d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)]/2, d(fx_n, Tx_n).d(fx_{n+1}, Tx_{n+1}), \\ &\quad d(fx_n, Tx_n).[d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)]/2, d(fx_{n+1}, Tx_{n+1}).[d(fx_n, Tx_{n+1}) \\ &\quad + d(fx_{n+1}, Tx_n)]/2, d(fx_n, Tx_{n+1}. d(fx_{n+1}, Tx_n)\}. \end{aligned}$$

For the sake of simplicity, let $d_n := d(fx_n, fx_{n+1})$.

Then the above inequality, after simplification, reduces to

$$d_{n+1}^2 \leq q^{1-k}. \max \{d_n^2, d_n d_{n+1}, d_n[d_n + d_{n+1}]/2, d_{n+1}[d_n + d_{n+1}]/2\}. \tag{3.1}$$

We remark that in the construction of sequences $\{x_n\}$ and $\{fx_n\}$, x_n (for each n) is not a coincidence point of T and f. This together with $Tx_n \neq Tx_{n+1}$ means that $fx_n \neq fx_{n+1}$. Indeed, if at any stage $fx_n = fx_{n+1}$ then

$fx_n \in Tx_n$, and x_n is a coincidence point of T and f. Therefore, according to our construction of the sequences, $d_n \neq 0$. Hence the inequality (3.1) implies one of the following:

$d^2_{n+1} \leq \lambda\, d^2_{n+1}$, where $\lambda = q^{1-k}$;

$d_{n+1} \leq \lambda\, d_n$;

$d_{n+1} \leq [\lambda/4 + \sqrt{}(\lambda^2/16 + \lambda/2)]d_n$;

$d_{n+1} \leq [\lambda/(2 - \lambda)]d_n$.

Consequently,

$$d_{n+1} \leq \max\ \{\sqrt{}\lambda, \lambda, [\lambda/4 + \sqrt{}\,(\lambda^2/16 + \lambda/2)], [\lambda/\,(2 - \lambda)]\}\ d_n.$$

This gives $d_{n+1} \leq \beta d_n$, when $\beta = [\lambda/4 + \sqrt{}(\lambda^2/16 + \lambda/2)]$. Notice that $0 < \beta < 1$. Hence $\{fx_n\}$ is a Cauchy sequence.

Now let $f(Y)$ be a complete subspace of Y. Then the sequence $\{fx_n\}$ has a limit $f(Y)$. Call it b. Hence there exists a point $z \in Y$ such that $fz = b$. Since $d(fz, Tz) \leq d(fz, fx_{n+1}) + H(Tx_n, Tz)$, applying (1.1) to the last term of this inequality and making $n \to \infty$, the inequality gives $d(fz, Tz) \leq \sqrt{}q.d(fz, Tz)$, and $fz \in Tz$, since $\sqrt{}q < 1$ and Tz is closed. This argument applies to the case when $T(Y)$ is a complete subspace of Y, since $T(Y) \subseteq f(Y)$.

This proves (I) and (II). To see (III), let $m > n$. Then,

$$d(fx_n, fx_m) \leq d(fx_n, fx_{n+1}) + \ldots + d(fx_{m+1}, fx_m)$$

$$\leq (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1}).\ d(fx_0, fx_1)$$

$< \beta^n/(1 - \beta).d(fx_0, fx_1)$.

Making $m \to \infty$, we get (III).

Let $\psi = \{\phi: R_+ \to R_+ | \phi$ is upper semicontinuous and nondecreasing$\}$. Then, in view of the proof of theorem 3 (above) and Rhoades *et al.* [18, Theorem 1], we have two generalizations of Theorem 3. We shall need the following definitions.

The following three definitions are due to Rhoades *et al.* [18] when $Y = X$.

DEFINITION 4. Let Y be an arbitrary nonempty set and (X, d) a metric space. If for a point $x_0 \in Y$ there exists a sequence $\{x_n\}$ in Y such that $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$, then $O(T, f; x_0) := \{fx_n : n = 1, 2, \ldots\}$ is the orbit for (T, f) at x_0. Further $O(T, f; x_0)$ is called a regular orbit for (T, f) if for each n,

$$d(fx_{n+1}, fx_{n+2}) \leq H(Tx_n, Tx_{n+1}).$$

We remark that the symbol for the orbit $O(T, f; x_0)$ is sometimes used for a sequence.

DEFINITION 5. A subspace $T(Y)$ or $f(Y)$ is called (T, f)-orbitally complete if every Cauchy sequence of the form $\{fx_{n_i} : fx_{n_i} \in Tx_{n_{i-1}}\}$ converges in X, where $\{x_n\}$ is a sequence in Y.

DEFINITION 6. If, for a point $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that the sequence $O(T, f; x_0)$ converges in X then X is called (T, f)-orbitally complete with respect to x_0 or simply $(T, f; x_0)$-orbitally complete. Further, an orbit $O(T, f; x_0)$ is said to be asymptotically regular if $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0$, where fx_n and fx_{n+1} are from $O(T, f; x_0)$.

THEOREM 7. Let Y be a nonempty set and (X, d) a metric space, $T : Y \to CL(X)$ and $f : Y \to X$ such that $T(Y) \subseteq f(Y)$ and the following hold:

$$H^2 (Tx, Ty) \leq \phi(m(x, y)) \text{ for all } x, y \in Y; \tag{7.1}$$

$$\phi(t) < qt \qquad \text{for each } t > 0, \tag{7.2}$$

for some $q \in (0, 1)$ and $\phi \in \psi$.

If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence point z (say). Further if $Y = X$ and the pair (T, f) are IT-commuting at z then T and f have a fixed point provided that $ffz = fz$.

THEOREM 8. Let Y be a nonempty set, (X, d) a metric space, $T: Y \to CL(X)$ and $f: Y \to Y$ such that

$$H^2(Tx, Ty) \leq \phi(M(x, y)), \tag{8.1}$$

where

$$\begin{aligned} M(x, y) := q.\max \{&d^2(fx, fy), d(fx, fy).d(fx, Tx), d(fx, fy).d(fy, Ty),\\ &d(fx, fy).d(fx, Ty), d(fx, fy).d(fy, Tx), d(fx, Tx).d(fy, Ty),\\ &d(fx, Tx).d(fx, Ty), d(fx, Tx).d(fy, Tx), d(fy, Ty).d(fx, Ty),\\ &d(fy, Ty).d(fy, Tx), d(fx, Ty).d(fy, Tx)\}. \end{aligned}$$

for all $x, y \in Y$.

$$\phi(t) < t \text{ for each } t > 0, \quad \phi \in \psi. \tag{8.2}$$

If, for some $x_0 \in Y$, there exists a sequence $\{x_n\} \subseteq X$ such that the orbit $O(T, f; x_0)$ is regular and asymptotically regular, and $T(X)$ or $f(X)$ is $O(T, f; x_0)$-orbitally complete, then T and f have a coincidence. Further, if $Y = X$ and the pair (T, f) is IT-commuting just at a coincidence point z (say) and $ffz = fz$, then fz is a common fixed point of T and f.

We remark that the completeness requirement in Theorem 3 may evidently be replaced by orbital completeness. Further, for the existence of a common fixed point of T and f, the requirement $ffz = fz$ can not be dropped from the above theorems (see, for instance, [17], [21], [22, page 194-195] and [24]). This means that, under either of the above theorems with $Y = X$, a coincidence point of T and f need not be their fixed point.

REFERENCES

1. Ismat Beg and Akbar Azam, Fixed points of asymptotically regular multivalued mappings, Internat, J. Math. Math. Sci. 15 (1992), 15-30.

2. R. Bhaskaran and P. V. Subrahmanyam, Common fixed points in metrically convex spaces, J. Math. Phys. Sci. 18 (5) (1984), 65-70.

3. H. W. Corley, Some hybrid fixed point theorems related to optimization, J. Math. Anal. Appl. 120 (1986), 528-532.

4. Lj. B. Ćirić, Fixed points for generalized multivalued contractions, Math. Vesnik, 9 (24) (1972), 265-272.

5. O. Hadžić, A coincidence theorem for multivalued mappings in metric spaces, Studia Univ. Babes-Bolyai Math. 26(1981), No. 4, 65-67.

6. S. Itoh and W. Takahashi, Single-valued mappings, multivalued mappings and fixed point theorems, J. Math. Anal. Appl. 59 (1977), 514-521.

7. J. Jachymski, On Reich's question concerning fixed points of multimap, Boll. Un. Mat. Ital. (7) 9-A (1995), 453-460.

8. G. Jungck and B. E. Rhoadeds, Fixed points for set-valued functions without continuity, Indian J. pure Appl. Math. 29 (1998), 227-238.

9. H. Kaneko, Single valued and multivalued f-contractions, Boll. Un. Mat. Ital. (6) 4-A (1985), 29-33.

10. H. Kaneko, A comparison of contractive conditions for multivalued mappings, Kobe J. Math. 3 (1986), 37- 45.

11. H. Kaneko and S. Seesa, Fixed point theorems for compatible multivalued and single valued mappings, Internat. J. Math. Sci. 12 (1989), 257-262.

12. C. Kulshrestha, Single-valued mappings, multi-valued mappings and fixed points theorems in metric spaces, Ph.D. Thesis (supervised by Prof. S. L. Singh), Garhwal University, Srinagar, 1983.

13. T. Kubiak, Two coincidence theorems for contractive type multivalued mappings, Studia Univ. Babes-Bolyai Math. 30 (1985), 65-68.

14. Z. Liu, F. Zhang and J. Mao, Common fixed points for compatible mappings of type (A), Bull. Malaysian Math. Soc. 22 (1999), 67-86.

15. S. N. Mishra, S. L. Singh and Rekha Talwar, Nonlinear hybrid contractions on Menger and Uniform spaces, Indian J. Pure Appl. Math. 25 (1994), 1039-1052.

16. S. B. Nadler, Jr., Multivalued contraction mappings, Pacific J. Math.30 (1969), 475-488.

17. S. A. Naimpally, S. L. Singh and J. H. M. Whitfield : Coincidence theorems for hybrid contractions, Math. Nachr. 127(1986), 177-180.

18. B. E. Rhoades, S. L. Singh and C. Kulshrestha, Coincidence theorems for some multivalued mappings, Internat. J. Math. Math. Sci. 7(1984), 429-434.

19. K. P. R Sastry, I. H. N. Rao and K. P. R. Rao, A fixed point theorem for mutimaps, Indian J. Phy. Natur. Sci. 3 (sec B), (1983), 1-4.

20. S. L. Singh, K. S. Ha and Y. J. Cho, Coincidence and fixed points of nonlinear hybrid contractions, Internat. J. Math. Math. Sci. 12 (1989), 247-256.

21. S. L. Singh and Kulshrestha, Coincidence theorem in metric spaces, Indian J. Phy. Natur. Sci. 2B (1982), 19-22.

22. S. L. Singh and S. N. Mishra, Nonlinear hybrid contractions, Journal of Natural & Physical Sciences 5-8 (1993), 191-206.

23. S. L. Singh and S. N. Mishra, On general hybrid contractions, J. Austral. Math. Soc. (Series A) 66 (1999), 244-254.

24. S. L. Singh and S. N. Mishra, Coincidences and fixed points of nonself hybrid contractions, J. Math. Anal. and Appl. 256 (2001), 486-497.

25. S. L. Singh and S. N. Mishra, Coincidences and fixed points of reciprocally continuous and compatible hybrid maps, Internat J. Math. Sci. 30 (2002), 627-635.

26. Dejun Tan, Zeqing Liu and Jong Kyu Kin, Common fixed points for compatible mappings of Type (P) in 2-metric spaces, Nonlinear Funct. Anal. & Appl., Vol. 8, NO. 2 (2003), 215-232.

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

खण्ड 18 अंक (2) 2004

(1) प्रकाशन स्थान : गुरुकुल कांगड़ी, विश्वविद्यालय, हरिद्वार
(2) प्रकाशन की अवधि : वर्ष में एक खण्ड अधिकतम दो अंक
(3) मुद्रक का नाम : चन्द्र किरण सैनी
राष्ट्रीयता : भारतीय
व पता : किरण ऑफसेट प्रिंटिंग प्रेस,
: निकट गुरुकुल कांगड़ी फार्मेसी,
: कनखल, हरिद्वार – 249404 फोन नं0 245975
(4) प्रकाशक का नाम : डॉ0 ए0के0 चोपड़ा
राष्ट्रीयता : भारतीय
व पता : कुलसचिव,
: गुरुकुल कांगड़ी विश्वविद्यालय,
: हरिद्वार – 249404
(5) प्रधान सम्पादक : डा0 वीरेन्द्र अरोड़ा
राष्ट्रीयता : भारतीय
व पता : गणित विभाग,
: गुरुकुल कांगड़ी विश्वविद्यालय,
: हरिद्वार – 249404
(6) प्रबन्ध सम्पादक : डा0 पी0पी0 पाठक
राष्ट्रीयता : भारतीय
व पता : भौतिकी विभाग,
: गुरुकुल कांगड़ी विश्वविद्यालय,
: हरिद्वार – 249404
(7) स्वामित्व : गुरुकुल कांगड़ी विश्वविद्यालय,
हरिद्वार – 249404

मैं ए0के0 चोपड़ा, कुलसचिव, गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार, घोषित करता हूँ कि उपरिलिखित तथ्य मेरी जानकारी के अनुसार सही हैं।

हस्ताक्षर
डॉ0 ए0के0 चोपड़ा
कुलसचिव

Journal of Natural & Physical Sciences, Vol. 18 (2) (2004)

CONTENTS

CHEMISTRY

47-54 R.D.KAUSHIK, YOGESH C. NAINWAL, VIRENDRA SINGH AND A.K. CHAUBEY : Kinetics and mechanism of Mn (II) catalysed periodate oxidation of o-chloro aniline in aetone-water medium

MATHEMATICS

1-6 B.G. PACHPATTE : A note on generalizations of midpoint inequality.

7-16 K.P.R. RAO AND N. SRINIVASA RAO : Common fixed point theorems for four self-maps on metric spaces by altering distances

21-34 ANITA SHARMA AND V.K. SHARMA : Methods for solving cubic equations

35-46 YOGITA BANA AND V.K. SHARMA : Construction of some important vedis from śulba sūtra

55-76 B.P. MISHRA & S.K. SINGH : Strong summability of functions based on (D,α,β) (C,l,m) summability methods

89-96 VIRENDRA ARORA AND NIDHI HANDA : Kātyāyna śulba sūtra

97-104 S.L. SINGH AND AMAL M. HASHIM : New coincidences and fixed points of reciprocally continuous and compatible hybrid maps

105-116 S.L. SINGH AND RITU ARORA : Coincidence and fixed point theorems for hybrid contractions

PHYSICS

17-20 VIJAI KUMAR, R.P. VATS AND P.P. PATHAK : Harmful bioeffects of high frequency electromagnetic fields.

77-88 N.SINGH, DEVENDRA SINGH, VIKAS MISHRA AND PRAKASH MISHRA : Effect of polarizability on nucleation phenomenon during ice glaciation due to external electric field

Printed at : Kiran Offset Printing Press, Kankhal-Hardwar, Phone : +91-1334-245975